

$$\begin{aligned}
 &= \frac{(m+1)^2 (m^2 + 4m + 4)}{4} \\
 &= \frac{(m+1)^2 (m+2)^2}{4} \\
 &= \frac{(m+1)^2 [(m+1) + 1]^2}{4}
 \end{aligned}$$

This is of the same form as the theorem that has been assumed, if m is replaced by $m+1$. Hence if the theorem is true for $n = m$, it is proved true for $n = m+1$.

But the theorem has been verified for $n = 1$.

$\therefore$ it must be true for $n = 2$.

Since it is true for $n = 2$, it must be true for $n = 3$, and so on, for any finite positive integral value of n .

4. Show that $1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \dots + (2n-1)(2n)$

$$= \frac{n(n+1)(4n-1)}{3}$$

First verify the result for $n = 1$.

$$\text{L. H. S.} = 1 \cdot 2 = 2$$

$$\text{R. H. S.} = \frac{1 \cdot 2 \cdot 3}{3} = 2.$$

$\therefore$ the result is verified for $n = 1$.

Assume the result for $n = m$, i.e. that

$$1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \dots + (2m-1)(2m)$$

$$= \frac{m(m+1)(4m-1)}{3}$$

Add the next term $(2m+1)(2m+2)$ to both sides,

$$1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \dots + (2m-1)(2m) + (2m+1)(2m+2)$$

$$= \frac{m(m+1)(4m-1)}{3} + (2m+1)(2m+2)$$

$$= (m+1) \left[\frac{m(4m-1)}{3} + 2(2m+1) \right]$$

$$= (m+1) \left[\frac{4m^2 + 11m + 6}{3} \right]$$

$$\begin{aligned}
 &= \frac{(m+1)(m+2)(4m+3)}{3} \\
 &= \frac{(m+1)[(m+1)+1][4(m+1)-1]}{3}
 \end{aligned}$$

This is of the same form as the result we have assumed if m is replaced by $m+1$. Hence if the result is true for $n=m$, then it is proved true for $n=m+1$.

But the result is verified for $n=1$. $\therefore$ the result must be true for $n=2$. Since it is true for $n=2$, it must be true for $n=3$ and similarly for $n=4, 5, 6$ and so on for any finite positive integral value of n .

Exercises

Prove by Mathematical Induction that

1 $1+3+5+\dots+(2n-1)=n^2$.

2 $1+5+9+\dots+(4n-3)=n(2n-1)$

3 $1\cdot3+3\cdot5+5\cdot7+\dots+(2n-1)(2n+1)$
 $=\frac{n(4n^2+6n-1)}{3}$

4 $2+2^2+2^3+\dots$ to n terms $=2(2^n-1)$

5 $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

6 $1\cdot2+2\cdot3+3\cdot4+\dots$ to n terms
 $=\frac{n(n+1)(n+2)}{3}$

7 $1^2+3^2+5^2+\dots$ to n terms $=\frac{n(4n^2-1)}{3}$

8 $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\dots$ to n terms
 $=\frac{n(n+1)(n+2)(n+3)}{4}$

9 $a+(a+d)+(a+2d)+\dots$ to n terms
 $=\frac{n}{2}[2a+(n-1)d]$

10 $a+ar+ar^2+\dots$ to n terms $=\frac{a(r^n-1)}{r-1}$

PERMUTATIONS

8 Let us suppose that there are 3 routes from Mangalore to Bangalore. If we wish to go from Mangalore to Bangalore, we can take any one of the 3 possible routes. The return journey can also be made by any one of the 3 routes. Thus the journey can be performed to and fro in one of 9 ways. It should be clearly noted that the two journeys have nothing in common; the second journey is performed after the first and is independent of it.

9 Similar considerations lead us to the following fundamental axiom, viz., **if a certain thing can be done in one of m ways and another thing independent of the first can afterwards be done in one of n ways, the two things can together be done in one of mn different ways.**

10 When n different things are given and when we arrange any r of them in a row in all possible ways, we are said to **permute** the n things taken r at a time and all such possible arrangements are called **permutations** of n things taken r at a time. Thus if the four letters a, b, c, d are required to be permuted 2 at a time, all possible arrangements are $ab, ac, ad, bc, bd, cd, ba, ca, da, cb, db$ and dc . The required number is 12.

11 To find the number of permutations of n things taken r at a time.

Let the n things be $a, b, c, \dots$

Let there be r blank spaces in a row. The first space can be filled up in one of n ways, since any one of the n things can be put there. After filling up the first space in one of these n ways, the second place can be filled up in any one of $n-1$ ways, as it can be filled up with one of the remaining $n-1$ things. By the axiom enunciated

above, the two places can be filled up in one of $n(n-1)$ ways.

Now $n-2$ things are left behind. The third place can therefore be filled up in any one of $n-2$ ways. The first 3 places can therefore be filled up in one of $n(n-1)(n-2)$ ways.

The same argument can be repeated till the r places are filled up and we find that the number of ways of filling up the r places is $n(n-1)(n-2)\dots$ to r factors i.e., $n(n-1)(n-2)\dots(n-r+1)$.

This is the number of permutations required and is represented by the symbol ${}_nP_r$.

Note. The number of permutations of n things taken all at a time i.e., ${}_nP_n = n(n-1)(n-2)\dots$ to n factors
 $= n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1$.

This product of the first n natural numbers is represented by the symbol $\frac{n!}{1}$ or $n!$ (factorial n).

With this notation ${}_nP_r = \frac{n!}{n-r!}$.

Worked Examples

1 How many numbers of 4 digits can be formed with the figures 1, 2, 4, 7, 8, 9?

The required number is ${}_6P_4 = 6 \times 5 \times 4 \times 3 = 360$.

2 Six papers are set in an examination, of which two are Mathematical. In how many different orders can the papers be arranged so that the two Mathematical papers are not consecutive?

If there is no condition imposed on the relative order of the papers, the number of arrangements will be $6!$. Of these, let us find out in how many the two Mathematical papers come together. Consider the two papers as one.

e.g., imagine them pinned together so that they cannot be separated. We have now 5 papers to be arranged in all possible ways. This can be done in $5!$ ways. But in each of these $5!$ arrangements the two Mathematical papers can be arranged among themselves in $2!$ ways.

$\therefore$ the number of arrangements in which the two papers come together = $5! \cdot 2!$.

$\therefore$ the number of ways in which they do not come together = $6! - 5! \cdot 2!$
 $= 720 - 240$
 $= 480$.

3 In how many ways can 3 boys and 5 girls be arranged in a row so that no two boys are together?

Since no two boys are to be together, first arrange the girls in a row. This can be done in $5!$ ways. Then in the 6 spaces between the girls (including the beginning and the end) put the 3 boys in all possible ways. This can be done in ${}_6P_3$ ways.

$\therefore$ the required number = $5! \times {}_6P_3$
 $= 120 \times 120$
 $= 14400$.

4 Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken (1) all at a time (2) 3 at a time.

(1) To find the required numbers, we have to fill up in all possible ways four blank spaces with the four digits 2, 3, 4, 5.

Consider the digits in the units' places in all these numbers. The digit 2 will occur as often as there are ways of filling up the remaining 3 places with the remaining 3 digits 3, 4, 5 and will therefore occur $3 \times 2 \times 1 = 6$ times. Similarly each of the other three digits will also occur 6 times in the units' places.

Therefore, on adding up all the digits in the units' places, we get $6 \times 2 + 6 \times 3 + 6 \times 4 + 6 \times 5 = 84$.

The total for the digits in the tens' places is also 84, and so on.

$$\begin{aligned}\therefore \text{the required total} &= 84 + 840 + 8400 + 84000 \\ &= 84(1 + 10 + 100 + 1000) \\ &= 84 \times 1111 \\ &= 93324.\end{aligned}$$

(2) Proceeding in the same way as before, we get,
the required total $= 84(1 + 10 + 100)$
 $= 84 \times 111$
 $= 9324.$

Exercises

- Find the values of ${}_5P_2$, ${}_{10}P_3$, ${}_6P_4$ and ${}_5P_4$.
Ans. 20, 720, 360, 120
- If ${}_nP_n = 5040$, find the value of n .
Ans. 7
- How many numbers greater than 2000 can be formed with the digits 1, 2, 3, 4, 5?
Ans. 216
- If the permutations of a, b, c, d, e taken all together be written down in alphabetical order as in the dictionary and numbered, what would be the number of the permutation $debac$?
Ans. 93
- How many numbers less than 4000 can be formed with the digits 5, 2, 3, 4, 0?
Ans. 116 (excluding 0)
- In a debating club, seven members, $A, B, C, \dots$ desire to take part in the debate. In how many ways can a list of speakers be arranged so that (a) A speaks immediately after B , (b) A speaks after B ?
Ans. (a) 720
(b) 2520
- Find the number of ways in which 5 boys and 3 girls may be arranged in a row so that (1) the girls may be

all together, (2) no two girls are together. Ans. (1) 4320
(2) 14400

8 If the letters of the word 'late' be permuted and the words so formed be arranged as in a dictionary, what will be the rank of that word?
Ans. 14

9 Five persons are to address a meeting. If a specified speaker is to speak before another specified speaker, find the number of ways in which this could be arranged. In how many of these arrangements will the first speaker come immediately before the second?
Ans. 60; 24

10 In how many permutations of n things taken r at a time will a given thing (1) appear first, (2) not appear at all, (3) appear? Ans. (1) ${}_{n-1}P_{r-1}$; (2) ${}_{n-1}P_r$; (3) $r \times {}_{n-1}P_{r-1}$.

11 How many words of 8 letters can be formed from a given 18, so that two particular letters always appear together?
Ans. ${}_{16}P_6 \times 7 \times 2$

12 A shelf contains 7 Tamil books, 4 Kannada books, and 5 English books. In how many ways can you arrange them so that (1) the Tamil books are together, (2) the Tamil books are together and the Kannada books are together, (3) no two English books are together, (4) books of the same language are together?
Ans. (1) $\boxed{10} \boxed{7}$, (2) $\boxed{7} \boxed{7} \boxed{4}$, (3) $\boxed{11} {}_{12}P_5$, (4) $\boxed{3} \boxed{4} \boxed{7} \boxed{5}$

13 Find the number of permutations of n things in which (1) two particular things are together, (2) no two of 3 particular things are together.
Ans. (1) $2 \boxed{n-1}$ (2) $(n-3)(n-4) \boxed{n-2}$

14 Show from first principles that ${}_nP_r = n \times {}_{n-1}P_{r-1}$. Hence derive the value of ${}_nP_r$.

15 Show that ${}_nP_r = (n-r+1) {}_nP_{r-1}$.

16 How many numbers of four digits each can be formed from the digits 1, 2, 3, 4, 0 no digit occurring more than once?
Ans. 96

17 A party of 10 persons consisting of 3 Muhammadans, 5 Hindus and 2 Christians is to be arranged in a row. Find the number of arrangements such that no Christian may occupy any place other than the first four places and no Muhammadan may occupy any place other than the first eight.

Ans. ${}_4P_2 \times {}_6P_3 \times 5$

18 How many even numbers of 4 digits can be formed with the digits, 2, 4, 5, 7, 8 no digit being used more than once in each number?

Ans. 72

19 In how many ways can the letters of the word 'mobile' be arranged so that the consonants always occupy odd places?

Ans. 36

20 Find the number of significant numbers which can be formed by using any number of the digits 0, 5, 6, 7, 8 but using each digit not more than once in each number.

Ans. 260

21 If ${}_{2n}P_3 = 2 \times {}_nP_4$, find n .

Ans. $n = 8$

22 In how many ways can the letters of the word 'TREASON' be arranged? How many arrangements begin with T? How many begin with T and end with N?

Ans. 5040; 720; 120

12 Circular Permutations

We have hitherto considered permutations of n things in a row, taken r at a time. These are called linear permutations. We shall now consider the arrangements of n things around a circle, taken all at a time. These are called circular permutations.

In arrangements around a circle, only relative order is required. We can fix any one of the n things in some place and arrange the remaining $n - 1$ things in the remaining $n - 1$ places. This can be done in $(n - 1)$

ways. Hence the number of circular permutations (round a circle) is $(n - 1)$.

If in these arrangements clockwise and anticlockwise directions are considered identical the number will be $\frac{1}{2}(n - 1)$.

Generally in the case of n persons arranged round a circle the number is taken to be $(n - 1)$, whereas in the case of flowers arranged in the form of a garland the number is taken to be $\frac{1}{2}(n - 1)$.

Exercises

1 In how many ways can a party of 9 persons be seated round a table if (1) two of them are to be together, (2) if two of them are not to be together?

Ans. (1) $2 \cdot 7!$ (2) $6 \cdot 7!$

2 In how many ways can 8 beads be strung together so as to form a necklace?

Ans. $\frac{1}{2} \cdot 7!$

3 In how many ways can (1) 9 persons be seated round a table, and (2) 9 flowers be strung together to form a garland?

Ans. (1) $8!$ (2) $\frac{1}{2} \cdot 8!$

4 In how many ways may a party of 4 ladies and 4 gentlemen be arranged at a round table, the ladies and gentlemen being seated alternately?

Ans. 144

COMBINATIONS

13 Given 4 things a, b, c, d , if we form groups of 3 things from the four things in all possible ways, we are said to form combinations of 4 things taken 3 at a time. In this, there is absolutely no reference to the relative order in which the 3 things are to be kept. The object now is merely to make a group, a selection or a bundle of 3 out of the 4 things in all possible ways. Thus the groups are

abc, bcd, cda, dab . Thus there are 4 combinations of 4 things taken 3 at a time. This is symbolically written, ${}_4C_3 = 4$.

14 To find the number of combinations of n things taken r at a time.

Let us first make selections of r things from the given n things in all possible ways. This can be done in ${}_nC_r$ ways.

Then let us arrange the r things in each of these combinations. Each combination will then give rise to r permutations.

$\therefore$ the total number of permutations that we get is ${}_nC_r \times r$.

This number must therefore be equal to ${}_nP_r$,

$$\text{i.e., } {}_nC_r \times r = {}_nP_r,$$

$$\text{i.e., } {}_nC_r = \frac{{}_nP_r}{r}$$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{r}$$

$$= \frac{n}{r} \frac{n-1}{n-r}$$

15 Definition of ${}_n C_0$.

$$\text{We know that } {}_nC_r = \frac{n}{r} \frac{n-1}{n-r}$$

In this put $r = n$.

$$\text{Then } {}_nC_n = \frac{n}{n} \frac{n-1}{n-n}$$

$$\text{i.e. } 1 = \frac{1}{0}$$

i.e. 0 is taken to be equal to 1.

16 To prove that ${}_nC_r = {}_nC_{n-r}$.

This is obvious, since every time a group of r things is made out of n things, a group of $n - r$ things is left behind.

Therefore there will be as many groups of n things taken r at a time as those taken $n - r$ at a time.

Note. Put $r = 0$. ${}_nC_0 = {}_nC_n = 1$.

17 To prove that ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

The R.H. side represents combinations of $n + 1$ things taken r at a time. These combinations can be divided into two classes, (1) those that contain a particular thing 'a', (2) those that do not contain 'a'.

(1) Those that contain 'a' are got by forming combinations of the remaining n things taken $r - 1$ at a time and adding the letter 'a' to each of these. Thus these are ${}_nC_{r-1}$ in number.

(2) Those that do not contain 'a' are got by forming combinations of the remaining n things taken r at a time. Thus these are ${}_nC_r$ in number.

$$\therefore {}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r.$$

Worked Examples

1 In how many ways can a committee of 6 men and 4 women be formed out of 15 men and 8 women?

6 men can be chosen from 15 men in ${}_{15}C_6$ ways. Thereafter 4 women can be chosen from 8 women in ${}_8C_4$ ways.

$\therefore$ the number of ways required

$$\begin{aligned} &= {}_{15}C_6 \times {}_8C_4 \\ &= \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \\ &= 350350. \end{aligned}$$

2 A father with 8 children takes them 3 at a time to the Zoological Gardens, as often as he can without taking the same 3 children together more than once. How often will he go, and how often will each child go?

He will go as often as he can choose 3 children from 8

$$\begin{aligned} \text{i.e., } {}_8C_3 \text{ times} &= \frac{8 \times 7 \times 6}{1 \times 2 \times 3} \\ &= 56 \text{ times.} \end{aligned}$$

Each child can go as often as he (or she) can be accompanied by two others i.e., ${}_7C_2$

$$\begin{aligned} &= \frac{7 \times 6}{1 \times 2} \\ &= 21 \text{ times.} \end{aligned}$$

3 How many diagonals can be drawn in a figure of n straight lines as sides?

There will be as many diagonals as there are ways of choosing 2 out of the n vertices, i.e., there will be ${}_nC_2$ diagonals. But this includes the lines joining adjacent vertices which are the sides of the figure.

$$\begin{aligned} \therefore \text{ the required number} &= {}_nC_2 - n \\ &= \frac{n(n-1)}{2} - n \\ &= \frac{n(n-3)}{2}. \end{aligned}$$

4 Out of 20 men who have to sit down, half on each side of a long table, 3 particular men desire to sit on one side and 4 others on the other side. Find the number of ways in which the party can be seated.

Out of the 20 men 7 have chosen sides for themselves. Therefore from the remaining 13, we have to choose 6 for one side of the table and then the remaining 7 will go over to the other side. The choice of 6 for one side can be made in ${}_{13}C_6$ ways.

We have now 10 men on each side. They can be arranged in 10 ways each.

$\therefore$ the total no. of ways in which they can all be seated $= {}_{13}C_6 \times \underline{10} \times \underline{10}$.

Exercises

1 Find the values of ${}_{10}C_3$, ${}_6C_2$, ${}_{14}C_{11}$ and ${}_{12}C_9$
Ans. 120, 15, 364, 220

2 If ${}_nC_{30} = {}_nC_5$, find n Ans. 35

3 If ${}_nC_5 = {}_nC_{17}$, find ${}_nC_{20}$ Ans. 231

4 If ${}_{15}C_{r+3} = {}_{15}C_{2r-3}$, find r Ans. 5; 6

5 4 scholarships are available for a school in which there are 3 Muhammadan pupils and 50 Hindu pupils. In how many ways can they be awarded, so as to include at least one Muhammadan? Ans. 62525.

6 A reserve of 12 Railway Station masters is to be divided into groups of 6 each, one for day duty and the other for night duty. In how many ways can this be done if two specified persons A and B should not be included in the same group? Ans. 504

7 From a council of 7 Europeans, 5 Hindus and 3 Muhammadans a committee of 5 is to be formed with one Muhammadan and at least two Europeans. Show that this can be done in 1260 ways.

8 There are n points in a plane, no three of which are collinear, except p which lie on one and the same straight line. How many lines can be drawn by joining pairs of these points and how many triangles can be formed by joining sets of 3 points?

$$\text{Ans. } {}_nC_2 - {}_pC_2 + 1, {}_nC_3 - {}_pC_3$$

9 If ${}_nC_3 = {}_nC_4 \times 24$, find n . Ans. 5

10 If ${}_nP_r = 240$, and ${}_nC_r = 120$, find n and r .
Ans. $n = 16$
 $r = 2$

11 If ${}_nP_4 = 24 \times {}_nC_5$, find n . Ans. $n = 9$

12 The results of 21 football matches (win, lose or draw) are to be predicted. How many different forecasts can contain exactly 18 correct results? Ans. 10640

13 What is the greatest number of points of inter-

section made by m straight lines and n circles?

Ans. $2mn + {}_mC_2 + 2 \cdot {}_nC_2$

14 Prove that $r \times {}_nC_r = (n - r + 1) {}_nC_{r-1}$.

15 3 scholarships of equal value are to be given to 10 candidates. How many different scholarship lists can be made? Ans. 120

16 How many different parties of r men can be formed out of n men (1) so as always to exclude p men, (2) so as always to include p men?

Ans. (1) ${}_{n-p}C_r$, (2) ${}_{n-p}C_{r-p}$

17 Find the number of combinations of n things $a, b, c, \dots$ taken r at a time in which (1) a and b occur, (2) a occurs but not b , (3) neither a nor b occurs, (4) a and b occur but not c .

Ans. (1) ${}_{n-2}C_{r-2}$, (2) ${}_{n-2}C_{r-1}$, (3) ${}_{n-2}C_r$, (4) ${}_{n-3}C_{r-2}$.

18 In how many ways can a party of 16 men be divided into two groups of 6 and 10? What is the number if two of them refuse to go in the same group?

Ans. ${}_{16}C_6, 2 \times {}_{14}C_5$

19 In how many ways can 8 oranges be divided between two persons? Ans. ${}_8C_1 + {}_8C_2 + \dots + {}_8C_7$

20 Out of 17 consonants and 5 vowels, how many words can be formed, each containing two consonants and one vowel? Ans. 4080

21 How many words of 2 consonants and one vowel can be formed from 6 consonants and 3 vowels, the vowel being the middle letter of each word? Ans. 90

22 26 people wish to travel by an omnibus which holds 12 inside and 14 outside. In how many ways can (1) the inside, (2) the outside passengers be selected? (3) In how many ways can they be selected if 8 refuse to ride outside and 6 will not go inside?

Ans. (1) ${}_{26}C_{12}$, (2) ${}_{26}C_{12}$, (3) 495

23 A railway carriage can seat 5 each side. Find in how many ways a party of 4 ladies and 6 gentlemen can

seat themselves so that the ladies may always have the corner seats? Ans. 17280

24 How many numbers are there consisting of 5 different digits arranged in ascending order of magnitude from left to right? Ans. 126

25 There are 12 intermediate stations on a Railway line between two stations. In how many ways can a train be made to stop at 4 of these intermediate stations, no two of these halting stations being consecutive? Ans. 126

26 Out of 6 masters and 4 boys, a committee of 8 is to be chosen. In how many ways can this be done, (1) when there are not less than 4 masters in the committee, (2) when there is a majority of masters?

Ans. (1) 45, (2) 30

27 Out of 7 caste Hindus, 3 Muslims and 2 Harijans, a cabinet of 8 ministers is to be formed, no community being left without a representative. In how many ways can this be done, if the number of Harijans is never to be less than the number of Muslims in the cabinet? Ans. 210

28 How many even numbers of 3 digits can be formed with the figures 2, 4, 5, 7, 8 no digit being used more than once in each number? Ans. 36

29 In how many of the number of permutations of n things, taken r at a time, will three given things occur?

Ans. $\frac{n-3}{n-r} \times r(r-1)(r-2)$

30 In how many ways can four Sanskrit and one Tamil book be placed on a shelf so that the Tamil book is always in the middle, the selection being made from seven Sanskrit and three Tamil books? Ans. 2520

31 From 7 gentlemen and 4 ladies a committee of 6 is to be formed. In how many ways can this be done when the committee contains (1) exactly 2 ladies and (2) at least 2 ladies? Ans. (1) 210, (2) 371

BINOMIAL THEOREM

18 To prove that if n is a positive integer, then $(x + a)^n = x^n + {}_nC_1 x^{n-1}a + {}_nC_2 x^{n-2}a^2 + \dots + {}_nC_r x^{n-r}a^r + \dots + a^n$.

Method of Induction

Let us first verify the theorem for a small elementary value of n , say $n = 2$.

$$\text{When } n = 2, \text{ L.H.S.} = (x + a)^2$$

$$\text{R.H.S.} = x^2 + 2xa + a^2.$$

Thus the two sides are equal and the theorem is verified for $n = 2$.

Next let us assume the theorem for $n = m$

$$\text{i.e., } (x + a)^m = x^m + {}_mC_1 x^{m-1}a + {}_mC_2 x^{m-2}a^2 + \dots$$

$$+ {}_mC_r x^{m-r}a^r + \dots + a^m.$$

Multiplying both sides by $x + a$, we get

$$(x + a)^{m+1} = (x^m + {}_mC_1 x^{m-1}a + {}_mC_2 x^{m-2}a^2 + \dots$$

$$+ {}_mC_r x^{m-r}a^r + \dots + a^m)(x + a)$$

$$= x^{m+1} + x^m a(1 + {}_mC_1) + x^{m-1}a^2({}_mC_1 + {}_mC_2) + \dots$$

$$+ x^{m-r+1}a^r({}_mC_{r-1} + {}_mC_r) + \dots + a^{m+1}$$

$$= x^{m+1} + {}_{m+1}C_1 x^m a + {}_{m+1}C_2 x^{m-1}a^2 + \dots$$

$$+ {}_{m+1}C_r x^{m+1-r}a^r + \dots + a^{m+1}, \therefore \text{ we know that}$$

$${}_mC_{r-1} + {}_mC_r = {}_{m+1}C_r.$$

This expression on the R.H. side is of the same form as the one we have assumed except that $m + 1$ is written for m .

$\therefore$ if the theorem is assumed true for $n = m$, it has been proved true for $n = m + 1$.

But we have verified the theorem for $n = 2$.

$\therefore$ it must be true for $n = 3$.

Since it is true for $n = 3$, it must be true for $n = 4$, and similarly for $n = 5$ and so on for any positive integral value of n .

19 It may be noted that

(1) the number of terms in the expansion is $n + 1$,

(2) the indices of 'x' go on decreasing and those of 'a' go on increasing by 1 at each stage so that the degree of any term is n .

(3) the coefficients are $1, {}_nC_1, {}_nC_2, {}_nC_3, \dots, {}_nC_{n-1}, {}_nC_n$ in order.

(4) the coefficients of terms equidistant from the beginning and the end are the same since ${}_nC_r = {}_nC_{n-r}$.

(5) the $(r + 1)^{\text{th}}$ term, $T_{r+1} = {}_nC_r x^{n-r}a^r$.

Worked Examples

1 Expand $(x + 3)^6$ in descending powers of x .

$$\begin{aligned} (x + 3)^6 &= x^6 + {}_6C_1 \cdot x^5 \cdot 3 + {}_6C_2 \cdot x^4 \cdot 3^2 + {}_6C_3 \cdot x^3 \cdot 3^3 \\ &\quad + {}_6C_4 \cdot x^2 \cdot 3^4 + {}_6C_5 \cdot x \cdot 3^5 + {}_6C_6 \cdot 3^6 \\ &= x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x \\ &\quad + 729. \end{aligned}$$

2 Expand $(2 - x)^5$.

$$\begin{aligned} (2 - x)^5 &= 2^5 + {}_5C_1 \cdot 2^4 \cdot (-x) + {}_5C_2 \cdot 2^3 \cdot (-x)^2 + {}_5C_3 \cdot 2^2 \cdot (-x)^3 \\ &\quad + {}_5C_4 \cdot 2 \cdot (-x)^4 + {}_5C_5 \cdot (-x)^5 \\ &= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5. \end{aligned}$$

3 Write down the 100th term of $(7 - 3x)^{125}$.

$$\text{General term } T_{r+1} = {}^{125}C_r \cdot 7^{125-r} \cdot (-3x)^r$$

$$\begin{aligned} \text{Putting } r = 99, T_{100} &= {}^{125}C_{99} \cdot 7^{26} \cdot (-3x)^{99} \\ &= -{}^{125}C_{26} \cdot 7^{26} \cdot 3^{99} \cdot x^{99}. \end{aligned}$$

4 Find the middle term in the expansion of $(2x + 1)^{18}$.

Since there are 19 terms in the expansion, the middle term is clearly the 10th.

$$\text{General term } T_{r+1} = {}^{18}C_r \cdot (2x)^{18-r} \cdot (1)^r.$$

$$\text{Putting } r = 9, T_{10} = {}^{18}C_9 (2x)^9.$$

Exercises

Expand the following

① $\left(3x + \frac{1}{x}\right)^5.$

$$2 \left(x - \frac{2}{x^2}\right)^4 \checkmark$$

$$3 \left(\frac{1}{\sqrt{x}} + 3x^2\right)^7 \checkmark$$

Find the two middle terms in the expansion of

$$④ (5x - 1)^3. \quad \text{Ans. } 1250x^3, -250x^2$$

$$5 \left(2 + \frac{1}{4x}\right)^7 \checkmark \quad \text{Ans. } \frac{35}{4x^3}, \frac{35}{32x^4}$$

Find the term independent of x in the expansion of

$$⑥ \left(x + \frac{1}{x}\right)^{18}. \quad \text{Ans. } {}_{18}C_9$$

$$⑦ \left(2x - \frac{3}{x^2}\right)^{15}. \quad \text{Ans. } -2^{10} \cdot 3^5 \cdot {}_{15}C_5$$

$$⑧ \left(x^2 - \frac{2}{x}\right)^{6n}. \quad \text{Ans. } 2^{4n} \cdot {}_{6n}C_{2n}$$

Find the coefficients of

$$⑨ x^6 \text{ in the expansion of } (2 - x)^7. \quad \text{Ans. } 14$$

$$10 x^4 \text{ in the expansion of } (4 + 3x)^6. \quad \text{Ans. } 19440$$

$$⑪ x^4 \text{ in the expansion of } \left(2x^2 - \frac{3}{x}\right)^8. \quad \text{Ans. } {}_8C_4 \times 16 \times 81$$

$$12 x^2 \text{ in the expansion of } (1 + x - x^2)^3. \quad \text{Ans. } 0$$

$$13 \text{ Find the term independent of } x \text{ in the expansion of } \left(\sqrt{x} - \frac{2}{x^2}\right)^{20}. \quad \text{Ans. } 16 \times {}_{20}C_4$$

$$14 \text{ Find the coefficient of } x^{11} \text{ in the expansion of } \left(x + \frac{2}{x^2}\right)^{17}. \quad \text{Ans. } 544$$

$$15 \text{ Find the coefficient of } x^{10} \text{ in the expansion of } (1 + x + x^2)(1 - x)^{15}. \quad \text{Ans. } 4433$$

$$16 \text{ Write down the middle terms in the expansion of } \left(x - \frac{1}{x}\right)^{15}. \quad \text{Ans. } -{}_{15}C_7 x, \frac{{}_{15}C_7}{x}$$

$$17 \text{ Find the coefficient of } x^4 \text{ in the expansion of } (1 + x - x^2)^{10}. \quad \text{Ans. } -105$$

18 Write down the coefficients of the r th, $(r + 1)$ th and $(r + 2)$ th terms in the expansion of $(1 + x)^n$. If these are in H.P., show that $n^2 - n(4r - 1) + 4r^2 = 0$.

19 The first three terms in the expansion of $(1 + ax)^n$ where n is a positive integer are $1 + 6x + 16x^2$. Find the values of a and n . Ans. $a = \frac{2}{3}, n = 9$

20 Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{x}\right)^{18}$ Ans. ${}_{18}C_6$

20 Binomial Coefficients

It is usual to denote the coefficients in the expansion of $(1 + x)^n$ by $c_0, c_1, c_2, \dots, c_n$ instead of $1, {}_nC_1, {}_nC_2, \dots, {}_nC_n$. In this form we have,

$$(1 + x)^n = c_0 + c_1x + c_2x^2 + \dots + c_r x^r + \dots + c_n x^n.$$

Putting $x = 1$ in the above, we get,

$$c_0 + c_1 + c_2 + \dots + c_n = 2^n.$$

Putting $x = -1$, we also get

$$c_0 - c_1 + c_2 - c_3 + \dots + (-1)^n c_n = 0.$$

$\therefore$ it follows that

$$\begin{aligned} c_0 + c_2 + c_4 + \dots &= c_1 + c_3 + \dots \\ &= \frac{1}{2}(c_0 + c_1 + c_2 + \dots + c_n) \\ &= \frac{1}{2} \times 2^n \\ &= 2^{n-1}. \end{aligned}$$

21 To sum the series $ac_0 + (a + d)c_1 + (a + 2d)c_2 + \dots + (a + nd)c_n$.

Let $S = ac_0 + (a + d)c_1 + (a + 2d)c_2 + \dots + (a + nd)c_n$.

Rewriting the series in the reverse order,

$$S = (a + nd)c_n + (a + n - 1d)c_{n-1} + \dots + ac_0.$$

Adding, since $c_0 = c_n, c_1 = c_{n-1}$ etc.,

$$\begin{aligned} 2S &= (2a + nd)(c_0 + c_1 + c_2 + \dots + c_n) \\ &= (2a + nd) \cdot 2^n. \end{aligned}$$

$$\therefore S = (2a + nd) \cdot 2^{n-1}.$$

Worked Examples

1 Show that $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \dots + \frac{c_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$.

$$\begin{aligned} \text{L.H.S.} &= 1 + \frac{n}{2} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{3} + \dots + \frac{1}{n+1} \\ &= \frac{1}{n+1} \left[(n+1) + \frac{(n+1)n}{1 \cdot 2} + \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3} + \dots + 1 \right] \\ &= \frac{1}{n+1} \left[{}_{n+1}C_1 + {}_{n+1}C_2 + \dots + {}_{n+1}C_{n+1} \right] \\ &= \frac{1}{n+1} [2^{n+1} - 1]. \end{aligned}$$

2 Show that $c_0 c_r + c_1 c_{r+1} + c_2 c_{r+2} + \dots + c_{n-r} c_n$

$$= \frac{{2n \choose n-r} {2n \choose n+r}}$$

We have

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \dots + c_r x^r + \dots + c_n x^n.$$

Rewriting the same series in the reverse order and making use of the result $c_r = c_{n-r}$.

$$(1+x)^n = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n.$$

Picking out the coefficient of x^{n+r} in the product of the two series on the R.H.S. we get

$$\begin{aligned} &c_0 c_r + c_1 c_{r+1} + c_2 c_{r+2} + \dots + c_{n-r} c_n. \\ \therefore \text{ the given series} &= \text{coefficient of } x^{n+r} \text{ in } (1+x)^{2n} \\ &= {}_{2n}C_{n+r} \\ &= \frac{{2n \choose n+r} {2n \choose n-r}}{1}. \end{aligned}$$

Exercises

Sum the following series:—

1 $c_0 + 2c_1 + 3c_2 + \dots + (n+1)c_n$. Ans. $2^{n-1}(n+2)$

2 $c_1 + 2c_2 + 3c_3 + \dots + nc_n$. Ans. $n \cdot 2^{n-1}$

3 $3c_1 + 5c_2 + \dots + (2n+1)c_n$ Ans. $(n+1)2^n - 1$

4 Show that $\frac{c_0}{1} + \frac{c_2}{3} + \frac{c_4}{5} + \dots = \frac{2^n}{n+1}$.

5 Show that $\frac{c_1}{c_0} + 2\frac{c_2}{c_1} + 3\frac{c_3}{c_2} + \dots$

$$+ n \cdot \frac{c_n}{c_{n-1}} = \frac{n(n+1)}{2}.$$

6 Prove that $(c_0 + c_1)(c_1 + c_2)(c_2 + c_3) \dots (c_{n-1} + c_n)$

$$= \frac{(n+1)^n}{n!} c_1 c_2 c_3 \dots c_n$$

7 Show that $c_1 - 2c_2 + 3c_3 - \dots + n(-1)^{n-1}c_n = 0$

Hence show that $c_1 + 3c_3 + 5c_5 + \dots$

$$= 2c_2 + 4c_4 + 6c_6 + \dots = n \cdot 2^{n-2}.$$

8 Sum the series:—

(1) $3c_0 + 7c_1 + 11c_2 + \dots$ to $(n+1)$ terms.

Ans. $(2n+3)2^n$

(2) $c_0 + 6c_1 + 11c_2 + \dots$ to $(n+1)$ terms.

Ans. $(5n+2)2^{n-1}$

(3) $5c_1 + 12c_2 + 19c_3 + \dots$ to n terms.

Ans. $(7n-4)2^{n-1} + 2$

9 Show that $c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2 = \frac{{2n \choose n}}{(n!)^2}$.

10 Show that $c_0 c_2 + c_1 c_3 + \dots + c_{n-2} c_n$

$$= \frac{{2n \choose n+2} {2n \choose n-2}}{1}.$$

11 If a_r be the coefficient of x^r in the expansion of $(1+x)^{2n}$, show that $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2$

$$= \frac{(-1)^n {2n \choose n}}{n! n!}.$$

ELEMENTARY PROPERTIES OF MATRICES
AND DETERMINANTS

22 Matrices

A Matrix is a rectangular array of numbers enclosed within brackets as follows: $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ The numbers $a_1, a_2, a_3, \dots$ are called elements of the matrix. A matrix is denoted by a capital letter such as A .

If there are m rows and n columns in a matrix it is called a $m \times n$ matrix or a matrix of order $m \times n$. The matrix given above as an example is a 2×3 matrix.

If in a matrix the number of rows and columns are the same and equal to n , it is called a square matrix of order n .

A matrix having only one row of n elements, such as $[a_1, a_2, a_3]$ is called a row matrix or row vector.

A matrix having only one column of n elements, such as $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is called a column matrix or column vector.

If the elements of a matrix are all zeroes it is called a null or zero matrix and is denoted by O ; e.g., $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$.

In a square matrix the elements in the left-to-right diagonal are called diagonal elements of the matrix;

e.g. in $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ a_1, b_2, c_3 are called diagonal elements.

If in a square matrix all the elements except the diagonal elements are zeroes it is called a diagonal matrix

e.g. $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

If in a diagonal matrix the diagonal elements are all equal it is called a scalar matrix. If all the diagonal elements are equal to 1, it is called a unit matrix and is denoted by I . (It is also called 'Identity matrix').

Thus $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are examples of scalar and unit matrices.

Note. A matrix has no numerical value.

23 Equality of Matrices

Two matrices are equal if they have the same order and also if corresponding elements are equal, that is, only if one is the exact duplicate of the other.

24 Operations on matrices

(a) Multiplication by a constant or scalar k .

If A is a given matrix, then the matrix kA is defined to be the matrix all of whose elements are k times the corresponding elements of A , i.e. if

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}, \text{ then } kA = \begin{bmatrix} ka_1 & ka_2 & ka_3 \\ kb_1 & kb_2 & kb_3 \end{bmatrix}.$$

$$\text{If } A = \begin{bmatrix} 4 & 1 & 3 & 5 \\ 2 & 0 & -1 & 4 \end{bmatrix}, \text{ then } 2A = \begin{bmatrix} 8 & 2 & 6 & 10 \\ 4 & 0 & -2 & 8 \end{bmatrix}.$$

(b) Addition of matrices

If two matrices are of the same order ($m \times n$) their sum is another matrix whose elements are the sum of corresponding elements of the 2 matrices;

$$\text{if } A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$\text{then } A + B = \begin{bmatrix} 2+1 & 1+5 & 3+3 \\ 5+2 & 2+3 & 4+0 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 6 \\ 7 & 5 & 4 \end{bmatrix}$$

Subtraction of one matrix from another can be similarly effected, e.g. in the above example,

$$A - B = \begin{bmatrix} 2-1 & 1-5 & 3-3 \\ 5-2 & 2-3 & 4-0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

Two matrices are said to be **conformable** or **comparable** for addition or subtraction if they are of the same order, $m \times n$. The sum or difference is also a matrix of the same order.

If A and B are matrices of different orders, then $A + B$ and $A - B$ are not defined.

Properties of Matrix addition. The following Laws are obviously true:

- If A , B and C are matrices conformable for addition, then
- (1) **Associative Law**: $A + (B + C) = (A + B) + C$
 - (2) **Commutative Law**: $A + B = B + A$
 - (3) **Distributive Law**: $k(A + B) = kA + kB$, k being a constant.

(c). Multiplication of Matrices

Two matrices of orders $m \times n$ and $n \times p$ are said to be conformable for multiplication. The rule for multiplication can be understood from the following:

$$\text{Let } A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \text{ and } B = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix}$$

A is a 2×3 matrix and B is a 3×2 matrix and therefore they are conformable for multiplication. The product is defined as follows:

$$AB = \begin{bmatrix} ap+br+ct & aq+bs+cu \\ dp+er+ft & dq+es+fu \end{bmatrix}$$

Thus AB is a 2×2 matrix.

$$\text{Similarly } \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \\ ep+fr & eq+fs \end{bmatrix}$$

The product of a 3×2 matrix with a 2×2 matrix is thus a 3×2 matrix.

As a numerical example let us take the following:

$$\text{Let } A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 4 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{Their product } AB &= \begin{bmatrix} 4 \times 1 + 3 \times 3 + 2 \times 4 & 4 \times 2 + 3 \times 3 + 2 \times 1 \\ 1 \times 1 + 2 \times 3 + 3 \times 4 & 1 \times 2 + 2 \times 3 + 3 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 19 \\ 19 & 11 \end{bmatrix} \end{aligned}$$

Thus the product of a 2×3 matrix with a 3×2 matrix is a 2×2 matrix.

In general, if A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$ then the product AB is a matrix of order $m \times p$.

Note. If A is $m \times n$ matrix and B is a $n \times p$ matrix, the product AB is defined but the product BA is not defined unless $p = m$.

If A is a square matrix of order n we write AA as A^2 , $A^2 A$ as A^3 and so on.

Worked Examples

- 1 Find BA in the example above.

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 8 \\ 15 & 15 & 15 \\ 17 & 14 & 11 \end{bmatrix}$$

Thus BA is a 3×3 square matrix.

Note also that $AB \neq BA$.

- 2 If $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ find A^2 .

$$A^2 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -3 \\ 5 & 8 & -3 \\ 6 & 3 & 4 \end{bmatrix}$$

Multiplication Laws of Matrices

1 Associative Law. If A, B, C , are conformable for multiplication then $A(BC) = (AB)C$

2 Distributive Law. $A(B + C) = AB + AC$
 $(B + C)A = BA + CA$, provided A, B, C are conformable for the operations involved.

Exercises

1 If $A = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$ find $A + B$ and $A - B$.

Ans. $\begin{bmatrix} 5 & 7 & 2 \\ 4 & 6 & 5 \end{bmatrix}; \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

2 If $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ find $2A + 3B$.

Ans. $\begin{bmatrix} 2 & 1 & 21 \\ 10 & 6 & 17 \end{bmatrix}$

3 If $A = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$ find AB .

Ans. $\begin{bmatrix} -3 & 6 & -6 \\ 7 & -14 & 14 \\ 1 & -2 & 2 \end{bmatrix}$

4 If $A = \begin{bmatrix} 2 & 3 & -1 \\ 2 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$ find AB .

Ans. $\begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix}$

5 If $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix}$ find A^2 .

Ans. $\begin{bmatrix} -5 & -2 & 5 \\ 2 & -1 & 7 \\ -6 & 2 & 0 \end{bmatrix}$

6 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ find AB and BA .

Ans. $\begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \\ 17 & 39 & 61 \end{bmatrix}; \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}$

7 If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ show that $A^2 = B^2 = I$.

25 Transpose of a matrix

If A is a $m \times n$ matrix and if the rows of A be changed into columns and the columns into rows we get a $n \times m$ matrix. This is called the transpose of A and is denoted by A' or A^T .

If $A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ then $A' = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 1 & 3 \end{bmatrix}$.

Exercises

1 If $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & -1 & 1 \end{bmatrix}$ find AA' and $A'A$.

Ans. $\begin{bmatrix} 13 & 3 \\ 3 & 11 \end{bmatrix}; \begin{bmatrix} 13 & 3 & 3 \\ 3 & 10 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

2 If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ find AB' and $A'B$.

Ans. $\begin{bmatrix} 5 & 15 \\ 10 & 20 \end{bmatrix}; \begin{bmatrix} 14 & 17 \\ 12 & 11 \end{bmatrix}$

3 Prove that $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy]$

4 Find $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Ans. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

5 Show that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 5 \\ 6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 5 \\ 6 & 2 & 3 \end{bmatrix}$

6 If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ show that $(AB)' = B'A'$.

$\begin{bmatrix} 11 & 5 \\ 18 & 8 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 5 & 8 \end{bmatrix}$

26 Determinants Let A be the square matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

This is a 2×2 matrix. We define the expression $a_1 b_2 - a_2 b_1$ to be the determinant of A and write

$$\det A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

For a 3×3 matrix we define the determinant in the following way: let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a 3×3 matrix.

We define the determinant of this matrix as

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Thus the elements of the first row, namely a_1, b_1, c_1 are multiplied respectively by $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ which have been defined already and the products added. This is the value of the determinant and can easily be seen to be $a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$.

The determinant got by suppressing the row and column containing an element is called the **minor** of that element. If the minors are denoted by capital letters, we can say that the value of the determinant of the third order is $a_1 A_1 - b_1 B_1 + c_1 C_1$.

27 Theorem The value of a determinant is unaltered if rows are changed into columns and columns into rows.

(1) Now $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$ and $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$.

Hence the theorem is true for a second order determinant.

(2) Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

From definition

$$\Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$

This can be rewritten as

$$\Delta = a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hence the theorem is true for the

third order determinant also.

Exercises

1 Evaluate the determinants

$$\begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix}; \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 5 & 4 & 3 \end{vmatrix}; \begin{vmatrix} 3 & 0 & -1 \\ -4 & 6 & 7 \\ 2 & -1 & -4 \end{vmatrix}$$

Ans. 1, 0, -43

2 Evaluate the determinants

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}; \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}; \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + a & 1 \\ 1 & 1 & 1 + b \end{vmatrix}$$

Ans. $3abc - a^3 - b^3 - c^3$; $abc + 2fgh - af^2 - bg^2 - ch^2$; ab .

28 Theorem If two rows or columns of a determinant are interchanged, the determinant is multiplied by -1 .

(1) Now $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$ and $\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = b_1 a_2 - b_2 a_1 = -(a_1 b_2 - a_2 b_1)$.

Hence $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$.

Similarly it can be shown that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$$

(2) Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \dots (1)$$

Let us interchange any two columns, say the first and second.

Let $\Sigma = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$

$$= b_1(a_2c_3 - a_3c_2) - a_1(b_2c_3 - b_3c_2) + c_1(b_2a_3 - b_3a_2).$$

$$= -\Delta, \text{ from (1) above.}$$

The same result is obtained if any other two columns are interchanged.

Similarly it can be shown that if any two rows are interchanged the determinant is multiplied by -1 .

Corollary

If in a determinant two rows or two columns are identical, the determinant vanishes.

For, if Δ is the value of the determinant, the value of the determinant got by interchanging the identical rows or columns is $-\Delta$. But the determinant is evidently the same as before.

$$\therefore \Delta = -\Delta \text{ i.e., } \Delta = 0.$$

29 Theorem If the elements of any row (or column) of a determinant are multiplied by K , the value of the determinant is multiplied by K .

Consider $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Sigma = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Now $\Sigma = Ka_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - Kb_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + Kc_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$$= K \left[a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \right]$$

$$= K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = K \Delta.$$

Note: A factor like K common to all the elements of a row (or column) can be removed outside the determinant.

Worked Example

Evaluate the determinant $\begin{vmatrix} 6 & 20 & 16 \\ 9 & 5 & 4 \\ 9 & 75 & 12 \end{vmatrix}$

Let the determinant $= \Delta$.

Then $\Delta = (3)(5)(4) \begin{vmatrix} 2 & 4 & 4 \\ 3 & 1 & 1 \\ 3 & 15 & 3 \end{vmatrix}$

$$= 60(2)(3) \begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 1 & 5 & 1 \end{vmatrix}$$

$$= 360 \{ 1(1-5) - 2(3-1) + 2(3-1) \}$$

$$= 360(20) = 7200.$$

Exercises

Evaluate the determinants:-

(1) $\begin{vmatrix} 8 & 14 & 41 \\ 4 & 7 & 35 \\ 12 & 21 & 73 \end{vmatrix}$, (2) $\begin{vmatrix} 2 & 3 & 5 \\ 6 & 6 & 10 \\ 10 & 9 & 15 \end{vmatrix}$

$$(3) \begin{vmatrix} 2 & 3 & 1 \\ 2 & 15 & 1 \\ 10 & 3 & 1 \end{vmatrix}, (4) \begin{vmatrix} 3 & 8 & 6 \\ 9 & 8 & 2 \\ 6 & 12 & 2 \end{vmatrix}. \text{ Ans. } 0, 0, -96, 288$$

30 Theorem. If each element in any row (or column) of a determinant consists of two terms, then the determinant can be expressed as the sum of two determinants.

$$\text{Thus } \begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{aligned} \text{For } & \begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= (a_1+x) \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (b_1+y) \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (c_1+z) \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &\quad + x \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - y \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + z \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

or, if
Corollary The determinant is —

$$\begin{aligned} \therefore \Delta &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &\quad + \begin{vmatrix} p & q & r \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \end{aligned}$$

Corollary II

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2+p & b_2+q & c_2+r \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p & q & r \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

31 Theorem If to the elements of any row (or column) of a determinant the same multiples of the corresponding elements of other rows (or columns) of the determinant are added, the value of the determinant is unaltered.

Consider the determinants

$$\begin{aligned} \Delta &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Sigma = \begin{vmatrix} a_1+ka_2+la_3 & b_1+kb_2+lb_3 & c_1+kc_2+lc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ \Sigma &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 & kb_2 & kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} la_3 & lb_3 & lc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= \Delta + k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + l \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= \Delta + k(0) + l(0) = \Delta. \end{aligned}$$

This theorem is very useful in simplifying and evaluating determinants.

Worked Examples

1 Evaluate $\Delta = \begin{vmatrix} p & q & r & 1 \\ q & r & p & 1 \\ r & p & q & 1 \end{vmatrix}$

Add the elements of the 1st column to corresponding elements of the 2nd column.

$$\text{Then } \Delta = \begin{vmatrix} p & (q+r)+p & 1 \\ q & (r+p)+q & 1 \\ r & (p+q)+r & 1 \end{vmatrix}$$

$$= (p+q+r) \begin{vmatrix} p & 1 & 1 \\ q & 1 & 1 \\ r & 1 & 1 \end{vmatrix} \begin{array}{l} \text{taking out the factor } p+q+r \\ \text{from each element of the 2nd} \\ \text{column,} \end{array}$$

$$= (p+q+r) \times 0 = 0.$$

2 Evaluate $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Let the determinant be $= \Delta$.

Then Δ is a homogeneous and symmetric expression in a, b, c of the 3rd degree.

In Δ put $a = b$. Then the determinant becomes

$$\begin{vmatrix} 1 & b & b^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\therefore a - b$ is a factor of Δ , by Remainder theorem.

Similarly, $b - c, c - a$ are factors of Δ .

$\therefore \Delta \equiv k(a-b)(b-c)(c-a)$, where k is a constant.

To find k put $a = 0, b = 1, c = 2$.

$$\text{Then } \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = k(-1)(-1)2$$

$$\text{i.e., } 2 = 2k \therefore k = 1 \therefore \Delta = (a-b)(b-c)(c-a).$$

or, if
termina
ns is -

as before, at

$$\therefore \Delta =$$

uate the determinant

Theorem
etermin

$$\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} \quad \text{Ans. } 1+a+b+c$$

Exercises

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

3 Evaluate $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ Ans. $3abc - a^3 - b^3 - c^3$

4 Evaluate $\begin{vmatrix} 8 & 15 & 7 \\ 20 & 36 & 17 \\ 5 & 9 & 4 \end{vmatrix}$ Ans. 3

5 Evaluate $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$ Ans. 0

6 Prove that $\begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix} = (x-p)(x-q)(x+p+q)$

7 Evaluate $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$ Ans. $(b-c)(c-a)(a-b)(a+b+c)$

8 Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

9 Show that $\begin{vmatrix} a-x & a-y & a-z \\ b-x & b-y & b-z \\ c-x & c-y & c-z \end{vmatrix} = 0$

10 Show that $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & y^2z & 0 \end{vmatrix} = 2x^3y^3z^3$

11 Evaluate $\begin{vmatrix} 1 & 3 & 6 \\ 3 & 10 & 22 \\ 3 & 11 & 26 \end{vmatrix}$ Ans. 0

12 Evaluate $\begin{vmatrix} 3 & 2 & -1 \\ 2 & 5 & 3 \\ -1 & 2 & 3 \end{vmatrix}$ Ans. $\sin^2 \frac{C}{2}$

13 Solve $\begin{vmatrix} 1 & 2 & 5 \\ 2 & x & 10 \\ 3 & 1 & -2 \end{vmatrix} = 0$ Ans. $\sin^2 \frac{C}{2}$

32 Solution of linear equations: Cramer's rule

(a) Solve the equations $a_1 x + b_1 y = c_1$; $a_2 x + b_2 y = c_2$.

Let $A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $A_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $A_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$\therefore Ax = \begin{vmatrix} a_1 x & b_1 \\ a_2 x & b_2 \end{vmatrix}$

Adding y times the second column to the first, we get

$Ax = \begin{vmatrix} a_1 x + b_1 y & b_1 \\ a_2 x + b_2 y & b_2 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = A_1$

$\therefore x = \frac{A_1}{A} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

Similarly $Ay = \begin{vmatrix} a_1 & b_1 y \\ a_2 & b_2 y \end{vmatrix}$

Adding x times the first column to the second, we get

$Ay = \begin{vmatrix} a_1 & a_1 x + b_1 y \\ a_2 & a_2 x + b_2 y \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = A_2$

$\therefore y = \frac{A_2}{A} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

This is known as Cramer's Rule for solving the equation, if

(termina the equations: $a_1 x + b_1 y + c_1 z = d_1$; $a_2 x + b_2 y + c_2 z = d_2$; $a_3 x + b_3 y + c_3 z = d_3$.

$\therefore \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $A_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $A_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

Theorem

termina

and $A_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$Ax = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$

Adding y times the second column and z times the third column to the first we get,

$Ax = \begin{vmatrix} a_1 x + b_1 y + c_1 z & b_1 & c_1 \\ a_2 x + b_2 y + c_2 z & b_2 & c_2 \\ a_3 x + b_3 y + c_3 z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = A_1$

$\therefore x = \frac{A_1}{A}$

Proceeding in a similar manner we can show that

$y = \frac{A_2}{A}$ and $z = \frac{A_3}{A}$.

This is Cramer's rule for solving equations in three variables.

Worked Examples

1 Solve $4x - 7y = 1$; $6x + y = 13$.

$A = \begin{vmatrix} 4 & -7 \\ 6 & 1 \end{vmatrix} = 46$, $A_1 = \begin{vmatrix} 1 & -7 \\ 13 & 1 \end{vmatrix} = 92$

and $A_2 = \begin{vmatrix} 4 & 1 \\ 6 & 13 \end{vmatrix} = 46$

$\therefore x = \frac{A_1}{A} = \frac{92}{46} = 2$, and $y = \frac{A_2}{A} = \frac{46}{46} = 1$

2 Solve: $4x - y + 2z = 9$, $3x + 2y - z = -1$, $2x + 3y + z = 1$.

$A = \begin{vmatrix} 4 & -1 & 2 \\ 3 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix} = 35$, on simplification

$A_1 = \begin{vmatrix} 9 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 3 & 1 \end{vmatrix} = 35$, $A_2 = \begin{vmatrix} 4 & 9 & \sin^2 \frac{C}{2} \\ 3 & -1 & \sin^2 \frac{C}{2} \\ 2 & 1 & \sin^2 \frac{C}{2} \end{vmatrix}$

$A_3 = \begin{vmatrix} 4 & -1 & 9 \\ 3 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix} = 70$

$$\therefore x = \frac{A_1}{A} = 1; \quad y = \frac{A_2}{A} = -1; \quad z = \frac{A_3}{A} = 2.$$

(The value of z can also be got by substituting the values of x and y in any of the given equations).

Exercises

Solve the following equations, using Determinants :

- 1) $3x + 10y = 16; 8x + 3y = 19.$
- 2) $5x - 3y + 9 = 0; 8x + 14y + 5 = 0.$
- 3) $x + y + z = 4; 2x + 5y - 2z = 3; 3x + 7y - 7z = 4.$
- 4) $2x + y + 3z = 11, x - 2y + z = 9, 3x + 2y + 2z = 8.$
- 5) $4x - y + 4z = 2, 2x + 3y + 5z = 2, 7x - 2y + 6z = 5.$
- 6) $4x + y = 7, 3y + 4z = 5, 3z + 5x = 2.$
- 7) $x + y + z = 7, 2x + 3y + 2z = 17, 4x + 9y + z = 37.$
- 8) $4x - 2y + 7z = -7, 6x + 4y - 3z = 10, 2x - y + 4z = -4.$

Ans. 1) $x=2, y=1$

3) $x=5, y=-\frac{9}{7}, z=\frac{2}{7}$

5) $x=3, y=2, z=-2$

7) $x=2, y=3, z=2$

2) $x=-\frac{3}{2}, y=\frac{1}{2}$

4) $x=2, y=-2, z=3.$

6) $x=1, y=3, z=-1.$

8) $x=\frac{1}{2}, y=1, z=-1.$

TRIGONOMETRY

CHAPTER I

PROPERTIES OF TRIANGLES

The angles and sides of a triangle are called the elements of the triangle. The angles are usually called A, B and C , and the sides opposite to these angles are denoted by a, b and c .

1 The Sine rule

In any $\triangle ABC$, to prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circumradius.

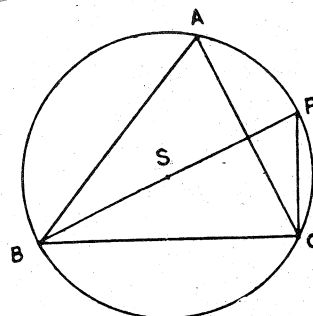


Fig. 1

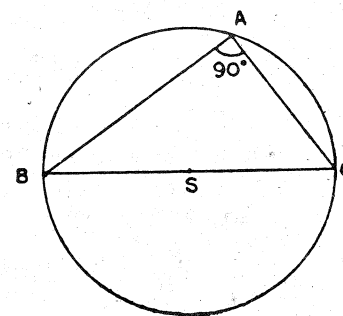
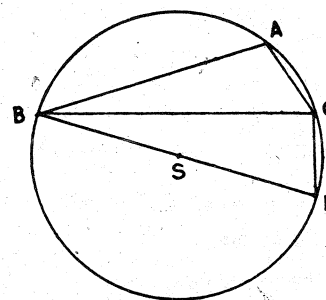


Fig. 2



$\sin^2 \frac{C}{2} = \frac{1 - \cos C}{2}$
 $\sin^2 \frac{C}{2} = \frac{1 - \cos C}{2}$
 $\sin^2 \frac{C}{2} = \frac{1 - \cos C}{2}$

Let us first prove that $\frac{a}{\sin A} = 2R$.

Case I. Let A be an acute angle as in fig. 1.

Draw the diameter BP . Join PC .

$$\angle BPC = \angle BAC = A.$$

Also $\angle BCP = 90^\circ$, $\therefore BP$ is a diameter.

$$\therefore \frac{BC}{BP} = \sin \angle BPC.$$

$$\text{i.e., } \frac{a}{2R} = \sin A, \text{ i.e., } \frac{a}{\sin A} = 2R.$$

Case II. Let $A = 90^\circ$, as in fig. 2.

Then BC is a diameter.

$$\text{i.e., } \frac{a}{\sin A} = 2R, \therefore \sin A = \sin 90^\circ = 1.$$

Case III. Let A be obtuse, as in fig. 3.

Draw the diameter BP .

$$\angle BPC = 180^\circ - \angle BAC = 180^\circ - A.$$

Also $\angle BCP = 90^\circ$.

$$\therefore \frac{BC}{BP} = \sin \angle BPC.$$

$$\text{i.e., } \frac{a}{2R} = \sin (180^\circ - A) = \sin A.$$

$$\text{i.e., } \frac{a}{\sin A} = 2R.$$

or, if $\frac{a}{\sin A} = 2R$ for all values of A , $\frac{a}{\sin A} = 2R$.
termina
is is — y $\frac{b}{\sin B} = 2R$, and $\frac{c}{\sin C} = 2R$.
ie as before

$$\therefore \Delta = \text{that } a = b \cos C + c \cos B$$

Draw $AD \perp BC$

Theorem Let C be acute, as in fig. 4.

$$\frac{BD}{AB} = \cos B, \text{ i.e., } BD = AB \cos B = c \cos B.$$

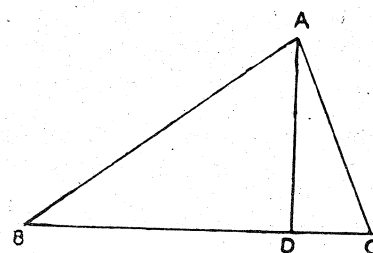


Fig. 4

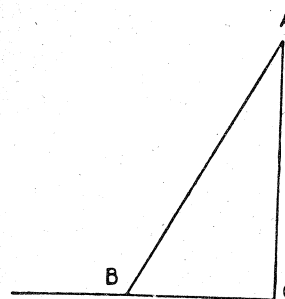


Fig. 5

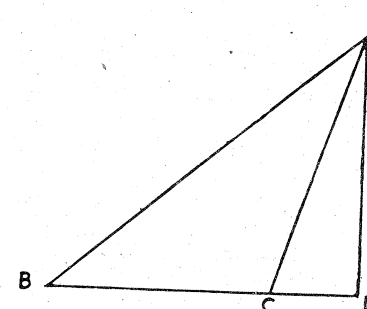


Fig. 6

Similarly $DC = b \cos C$.

But $BC = BD + DC$

$$\text{i.e., } a = c \cos B + b \cos C.$$

Case II. Let $C = 90^\circ$, as in fig. 5.

$$\frac{BC}{AB} = \cos B, \text{ i.e., } BC = AB \cos B$$

$$\text{i.e., } a = c \cos B$$

$$= c \cos B + b \cos C \therefore \cos C =$$

Case III. Let C be obtuse, as in fig. 6.

In this case, $BD = c \cos B$, as before.

$$\begin{aligned} DC &= AC \cos \angle ACD \\ &= AC \cos (180^\circ - C) \\ &= -b \cos C. \end{aligned}$$

$$BC = BD - DC$$

i.e., $a = c \cos B + b \cos C$
 $\therefore$ in all cases $a = b \cos C + c \cos B$.

Note: Similarly $b = c \cos A + a \cos C$ and
 $c = a \cos B + b \cos A$.

- 3 To prove that $a^2 = b^2 + c^2 - 2bc \cos A$.
 Draw $CD \perp r$ to AB .

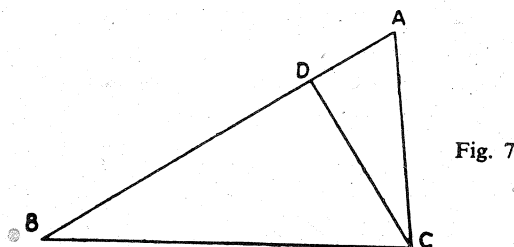


Fig. 7

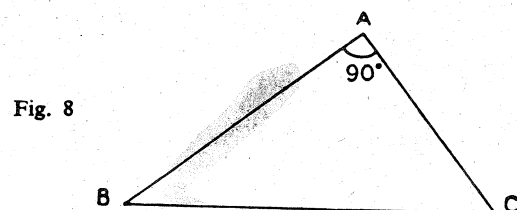


Fig. 8

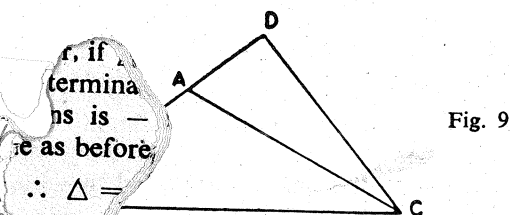


Fig. 9

Theorem Let A be acute, as in fig. 7.
 In Geometry, we have
 $= AB^2 + AC^2 - 2 AB \cdot AD$.

$$\frac{AD}{AC} = \cos A, \text{ i.e., } AD = b \cos A.$$

$$\therefore a^2 = c^2 + b^2 - 2c \cdot b \cos A = b^2 + c^2 - 2bc \cos A.$$

Case II. Let $A = 90^\circ$, as in fig. 8.

$$\text{In this case, } BC^2 = AB^2 + AC^2$$

$$\text{i.e., } a^2 = c^2 + b^2$$

$$= b^2 + c^2 - 2bc \cos A \quad \because \cos A = \cos 90^\circ = 0.$$

Case III. Let A be obtuse, as in fig. 9.

From Geometry, we have

$$BC^2 = AB^2 + AC^2 + 2AB \cdot AD.$$

$$\frac{AD}{AC} = \cos \angle DAC = \cos (180^\circ - A) = -\cos A.$$

$$\therefore AD = -b \cos A.$$

$$\therefore a^2 = c^2 + b^2 + 2c(-b \cos A)$$

$$= b^2 + c^2 - 2bc \cos A.$$

$$\therefore \text{in all cases, } a^2 = b^2 + c^2 - 2bc \cos A.$$

Note: (1) Similarly, $b^2 = c^2 + a^2 - 2ca \cos B$ and
 $c^2 = a^2 + b^2 - 2ab \cos C$.

$$(2) \text{ The same formulae are written sometimes as } \cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

(3) The results can also be derived from Art. 2.

$$a = b \cos C + c \cos B$$

$$\therefore a^2 = ab \cos C + ac \cos B$$

$$\text{Similarly } b^2 = bc \cos A + ba \cos C$$

$$\text{and } c^2 = ca \cos B + cb \cos A.$$

$$\therefore b^2 + c^2 - a^2 = 2bc \cos A.$$

- 4 To prove that in any $\triangle ABC$,

$$(1) \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \sin^2 \frac{C}{2} \text{ value}$$

$$(2) \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \sin^2 \frac{C}{2}$$

$$(3) \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \tan^2 \frac{C}{2}$$

We have shown that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\begin{aligned} (1) \quad \text{Now } 2 \sin^2 \frac{A}{2} &= 1 - \cos A \\ &= 1 - \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc - b^2 - c^2 + a^2}{2bc} \\ &= \frac{a^2 - (b - c)^2}{2bc} \\ &= \frac{(a + b - c)(a - b + c)}{2bc} \end{aligned}$$

If $2s = a + b + c$,

$$\begin{aligned} 2 \sin^2 \frac{A}{2} &= \frac{2(s - c) 2(s - b)}{2bc} \\ \therefore \sin^2 \frac{A}{2} &= \frac{(s - b)(s - c)}{bc} \\ \therefore \sin \frac{A}{2} &= \pm \sqrt{\frac{(s - b)(s - c)}{bc}} \end{aligned}$$

Since $\frac{A}{2}$ is an acute angle always, $\sin \frac{A}{2}$ is +ve

$$\text{Hence } \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}$$

$$\begin{aligned} (2) \quad \text{Now } 2 \cos^2 \frac{A}{2} &= 1 + \cos A \\ &= 1 + \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b + c)^2 - a^2}{2bc} \\ &= \frac{(b + c + a)(b + c - a)}{2bc} \\ &= \frac{2s \cdot 2(s - a)}{2bc} \end{aligned}$$

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$$\begin{aligned} \therefore \cos^2 \frac{A}{2} &= \frac{s(s - a)}{bc} \\ \therefore \cos \frac{A}{2} &= \pm \sqrt{\frac{s(s - a)}{bc}} \end{aligned}$$

Since $\frac{A}{2}$ is an acute angle always, $\cos \frac{A}{2}$ is +ve.

$$\therefore \cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}$$

$$(3) \quad \text{Now } \tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}$$

Or the value of $\tan \frac{A}{2}$ can be got independently as follows:-

$$\begin{aligned} \tan^2 \frac{A}{2} &= \frac{2 \sin^2 \frac{A}{2}}{2 \cos^2 \frac{A}{2}} = \frac{1 - \cos A}{1 + \cos A} \\ &= \frac{1 - \frac{b^2 + c^2 - a^2}{2bc}}{1 + \frac{b^2 + c^2 - a^2}{2bc}} \\ &= \frac{2bc - b^2 - c^2 + a^2}{2bc + b^2 + c^2 - a^2} \\ &= \frac{a^2 - (b - c)^2}{(b + c)^2 - a^2} \\ &= \frac{(a + b - c)(a - b + c)}{(a + b + c)(a - b - c)} \\ &= \frac{2(s - c) 2(s - b)}{2s \cdot 2(s - a)} \sin^2 \frac{C}{2} \text{ value } \frac{1}{2} \\ &= \frac{(s - b)(s - c)}{s(s - a)} \sin^2 \frac{C}{2} \\ \therefore \tan \frac{A}{2} &= \pm \sqrt{\frac{(s - b)(s - c)}{s(s - a)}} \end{aligned}$$

Since $\frac{A}{2}$ is an acute angle always, $\tan \frac{A}{2}$ is +ve

$$\therefore \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Note: In a similar way the values of the trigonometric ratios of $\frac{B}{2}$ and $\frac{C}{2}$ can be written down.

5 To prove that in any triangle ABC ,

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

$$\text{Now } \frac{b-c}{b+c} = \frac{2R \sin B - 2R \sin C}{2R \sin B + 2R \sin C}$$

$$= \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}$$

$$= \cot \frac{B+C}{2} \tan \frac{B-C}{2}$$

$$= \tan \frac{A}{2} \tan \frac{B-C}{2}$$

$$\therefore \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cdot \frac{1}{\tan \frac{A}{2}}$$

$$= \frac{b-c}{b+c} \cot \frac{A}{2}.$$

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$\therefore \Delta$ two other results can be obtained similarly:

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} \text{ and}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

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Worked Examples

1 Show that $\frac{\sin(B-C)}{\sin A} = \frac{b \cos C - c \cos B}{b \cos C + c \cos B}$

$$\text{L.H.S.} = \frac{\sin(B-C)}{\sin(B+C)}, \because B+C=180^\circ-A$$

$$= \frac{\sin B \cos C - \cos B \sin C}{\sin B \cos C + \cos B \sin C},$$

$$= \frac{2R \sin B \cos C - 2R \cos B \sin C}{2R \sin B \cos C + 2R \cos B \sin C},$$

multiplying by $2R$

$$= \frac{b \cos C - c \cos B}{b \cos C + c \cos B}, \because 2R \sin B = b, \text{ etc.}$$

2 Show that $a \cos \frac{B-C}{2} = (b+c) \sin \frac{A}{2}$

$$\frac{a}{b+c} = \frac{2R \sin A}{2R \sin B + 2R \sin C} = \frac{\sin A}{\sin B + \sin C}$$

$$= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}$$

$$= \frac{\sin \frac{A}{2}}{\cos \frac{B-C}{2}} \because \sin \frac{B+C}{2} = \cos \frac{A}{2}$$

$$\therefore a \cos \frac{B-C}{2} = (b+c) \sin \frac{A}{2}.$$

3 Show that $c^2 = (a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2}$

$$\text{R.H.S.} = (a^2 - 2ab + b^2) \cos^2 \frac{C}{2} + (a^2 + 2ab + b^2) \sin^2 \frac{C}{2}$$

$$= a^2 \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) + b^2 \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right)$$

$$\begin{aligned}
 & -2ab \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right) \\
 & = a^2 + b^2 - 2ab \cos C \\
 & = c^2.
 \end{aligned}$$

4 Show that $\sum \frac{b^2 - c^2}{a^2} \sin 2A = 0$.

$$\begin{aligned}
 \text{L.H.S.} &= \sum \frac{\sin^2 B - \sin^2 C}{\sin^2 A} 2 \sin A \cos A \\
 &= \sum \frac{\sin(B+C) \sin(B-C)}{\sin A} 2 \cos A \\
 &= \sum 2 \sin(B-C) \cos A, \because \sin(B+C) = \sin A \\
 &= -\sum 2 \sin(B-C) \cos(B+C) \\
 &\quad \because \cos A = -\cos(B+C) \\
 &= -\sum (\sin 2B - \sin 2C) \\
 &= 0.
 \end{aligned}$$

Exercises

Prove that

$$\begin{aligned}
 1 \quad & \frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2 - c^2}{a^2} \quad [\text{use the result}] \\
 & \sin^2 B - \sin^2 C = \sin(B+C) \sin(B-C). \\
 2 \quad & a^2 + b^2 + c^2 = 2(bc \cos A + ca \cos B + ab \cos C).
 \end{aligned}$$

$$3 \quad \sum a \sin(B-C) = 0.$$

$$4 \quad \sum \frac{a^2 \sin(B-C)}{\sin A} = 0.$$

$$5 \quad \sum a(b^2 + c^2) \cos A = 3abc.$$

[Hint: Rearrange and use Art. 2].

$$6 \quad \sum a^2 (\cot B - \cot C) = 0.$$

$$7 \quad \frac{b+c}{b-c} = \tan \frac{B+C}{2} \cot \frac{B-C}{2}.$$

$$8 \quad a \sin \left(\frac{A}{2} + B \right) = (b+c) \sin \frac{A}{2}.$$

$$9 \quad (a^2 - b^2 + c^2) \tan B = (a^2 + b^2 - c^2) \tan C.$$

10 If a^2 , b^2 and c^2 are in A.P., prove that $\tan A$, $\tan B$ and $\tan C$ are in H.P.

11 If $C = 60^\circ$, show that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$.

12 Prove that $\sum (b+c) \cos A = \sum a$.

13 If $\tan \theta = \frac{2\sqrt{bc}}{b-c} \sin \frac{A}{2}$ prove that $a \cos \theta = b - c$.

14 If $a \cos A = b \cos B$, show that the triangle is either isosceles or right-angled.

15 If $\cos A = \sin B - \cos C$, show that one of the angles is a right angle.

16 If $\sin^2 B + \sin^2 C = \sin^2 A$, then $A = 90^\circ$.

17 If $\sin 2B + \sin 2C = \sin 2A$, then the Δ is right-angled.

18 If $a = 2b$, and $A = 3B$, prove that $C = 60^\circ$.
Prove that

19 $a(\cos A + \cos B \cos C) = b(\cos B + \cos C \cos A) = c(\cos C + \cos A \cos B)$.

20 $(a+b+c) \sin \frac{A}{2} = 2a \cos \frac{B}{2} \cos \frac{C}{2}$.

21 If a, b, c are in A.P., show that $\tan \frac{A}{2} \tan \frac{C}{2} = \frac{1}{3}$.

22 If $3a = b + c$, prove that $\sin \frac{A}{2} = \sin \frac{B}{2} \sin \frac{C}{2}$.

23 If $C = 2B$, then $c^2 = ab + b^2$.

24 Prove that $\frac{a}{bc} + \frac{\cos A}{a} = \frac{b}{ca} + \frac{\cos B}{b} = \frac{c}{ab} + \frac{\cos C}{c}$.

25 Show that $\frac{\cot B - \cot C}{b^2 - c^2} = \frac{\cot C - \cot A}{c^2 - a^2} = \frac{\cot A - \cot B}{a^2 - b^2}$.

26 Show that

$$a = (b+c) \sin \phi, \text{ where } \cos \phi = \frac{2\sqrt{bc}}{b+c} \cos \frac{A}{2}.$$

27 $\sum a^3 \cos(B-C) = 3abc$.

✓ 28 Prove that $a^2 = (b - c)^2 \cos^2 \frac{A}{2} \sec^2 \phi$ where $\tan \phi = \frac{b + c}{b - c} \tan \frac{A}{2}$.

29 Prove that $b^2 + c^2 - 2bc \cos (60^\circ + A) = c^2 + a^2 - 2ca \cos (60^\circ + B) = a^2 + b^2 - 2ab \cos (60^\circ + C)$.

✓ 30 Show that $c \sin^2 \frac{B}{2} + b \sin^2 \frac{C}{2} = s - a$.

✓ 31 Show that $\frac{1}{a} \cos^2 \frac{A}{2} + \frac{1}{b} \cos^2 \frac{B}{2} + \frac{1}{c} \cos^2 \frac{C}{2} = \frac{s^2}{abc}$.

✓ 32 Prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$.

✓ 33 If a, b, c are in H.P., show that $\sin^2 \frac{A}{2}, \sin^2 \frac{B}{2}, \sin^2 \frac{C}{2}$ are also in H.P.

✓ 34 If a, b, c are in A.P., show that $\tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2}$ are in H.P.

35 In any triangle prove that $(a + b + c) \left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) = 2c \cot \frac{C}{2}$.

✓ 36 In a triangle ABC , $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$. Show that a, b, c are in A.P.

✓ 37 In a triangle ABC , $\tan \frac{A}{2} = \frac{5}{6}, \tan \frac{B}{2} = \frac{20}{37}$. Find $\tan \frac{C}{2}$ and show that a, b, c are in A.P.

SOLUTION OF TRIANGLES

6 Every triangle has six elements, three angles and three sides. Since the sum of three angles of a triangle is two right angles, if two angles are given, the third can be found out at once. Therefore the three angles of a triangle are not independent elements. If any three independent elements of a triangle are given the remaining elements can be calculated from the Trigonometric formulæ connecting the sides and angles of a triangle. The method of calculating the unknown elements is known as the solution of the triangle.

The problems on solution of a triangle can be grouped under four heads according to the elements that are given. They are

- (1) the three sides
- (2) two sides and the included angle
- (3) two sides and an angle opposite to one of them
- (4) one side and two angles.

We shall illustrate each of these with examples fully worked out.

7 Solution of the triangle when the three sides are given

The formula $a^2 = b^2 + c^2 - 2bc \cos A$ can be used to find out the value of angle A . Similarly the formula $b^2 = c^2 + a^2 - 2ca \cos B$ can be used to find out the value of angle B . The value of the third angle can then be written down since we know that $A + B + C = 180^\circ$.

Worked Example

Solve the triangle ABC given that $a=2$, $b=\sqrt{6}$, $c=\sqrt{3}+1$.

Apply the formula $a^2 = b^2 + c^2 - 2bc \cos A$.

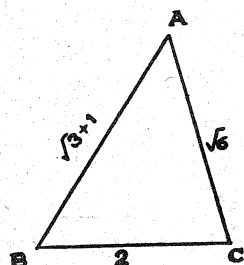


Fig. 10

$$4 = 6 + (\sqrt{3}+1)^2 - 2\sqrt{6}(\sqrt{3}+1)\cos A$$

$$\therefore \cos A = \frac{6 + (\sqrt{3}+1)^2 - 4}{2\sqrt{6}(\sqrt{3}+1)} = \frac{6 + 2\sqrt{3}}{2\sqrt{6}(\sqrt{3}+1)}$$

$$= \frac{2\sqrt{3}(\sqrt{3}+1)}{2\sqrt{6}(\sqrt{3}+1)} = \frac{1}{\sqrt{2}}$$

$$\therefore A = 45^\circ.$$

$$\text{Similarly } \cos B = \frac{(\sqrt{3}+1)^2 + 4 - 6}{2(\sqrt{3}+1)2} = \frac{2 + 2\sqrt{3}}{2(\sqrt{3}+1)2} = \frac{1}{2}$$

$$\therefore B = 60^\circ$$

$$\therefore \text{the remaining angle } C = 75^\circ.$$

8 Solution of the triangle when two sides and the included angle are given:

Suppose b , c and angle A are given. Let $b > c$.

Then take the formula $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$.

The R.H.S. is a known quantity.

$\therefore \frac{B-C}{2}$ can be found out.

But $\frac{B+C}{2} = 90^\circ - \frac{A}{2}$ and is therefore known.

Hence B and C can be found out.

Finally from the Sine Rule, $\frac{a}{\sin A} = \frac{b}{\sin B}$, the side a can be found out. This completes the solution.

Worked Example

Solve the triangle ABC given $b=2$, $c=\sqrt{3}+1$, and $A=30^\circ$.

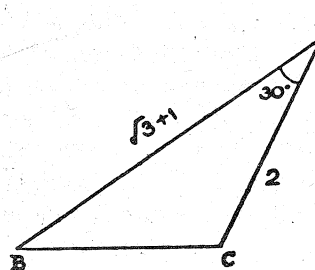


Fig. 11

Here $c > b$, hence $C > B$.

$$\therefore \tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2}$$

$$= \frac{\sqrt{3}+1-2}{\sqrt{3}+1+2} \cot 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+3} \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}(1+\sqrt{3})} = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{C-B}{2} = 30^\circ$$

$$\text{But } \frac{C+B}{2} = 90^\circ - \frac{A}{2} = 75^\circ$$

$$\therefore \text{adding, } C = 105^\circ$$

and subtracting, $B = 45^\circ$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\therefore a = \frac{b \sin A}{\sin B} = \frac{2 \cdot \sin 30^\circ}{\sin 45^\circ} = \frac{2 \cdot \frac{1}{2}}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

$\therefore$ solution is $a = \sqrt{2}$, $B = 45^\circ$, $C = 105^\circ$.

9 Solution of the triangle when two sides and an angle opposite to one of them are given (Ambiguous Case).

Suppose b, c and angle C are given.

$$\text{Then } \frac{\sin B}{b} = \frac{\sin C}{c}, \therefore \sin B = \frac{b \sin C}{c}$$

$\therefore B$ can be found out. This will have two values which are supplementary. For each of these values the corresponding value of angle A can be found out since $A + B + C = 180^\circ$. Then the corresponding value of a can be found out in each case from the Sine Rule. Thus there may be two solutions to the problem. In some cases one value of angle B will be found to be inadmissible.

Worked Examples

1 Solve the triangle ABC , given that $A=45^\circ$, $b=\sqrt{6}$, $a=2$.

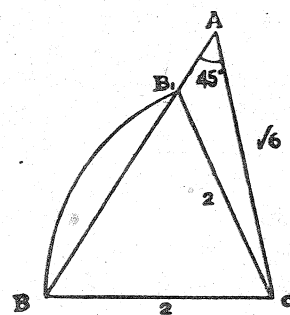


Fig. 12

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a} = \frac{\sqrt{6} \sin 45^\circ}{2} = \frac{\sqrt{6}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$B = 60^\circ \text{ or } 120^\circ.$$

Case I. Let $B = 60^\circ$

$$\text{Then } C = 180^\circ - 45^\circ - 60^\circ = 75^\circ$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\therefore c = \frac{a \sin C}{\sin A} = \frac{2 \sin 75^\circ}{\sin 45^\circ} = \frac{2(\sqrt{3}+1)\sqrt{2}}{2\sqrt{2}} = \sqrt{3}+1.$$

Case II. Let $B = 120^\circ$

$$\text{Then } C = 180^\circ - 45^\circ - 120^\circ = 15^\circ$$

$$\text{Then } c = \frac{a \sin C}{\sin A} = \frac{2 \sin 15^\circ}{\sin 45^\circ} = \frac{2(\sqrt{3}-1)}{2\sqrt{2}} \cdot \sqrt{2} = \sqrt{3}-1$$

$$\therefore \text{ solutions are } 1) B = 60^\circ, C = 75^\circ, c = \sqrt{3}+1$$

$$2) B = 120^\circ, C = 15^\circ, c = \sqrt{3}-1.$$

2 Solve the triangle ABC , given $a=2$, $B=120^\circ$, $b=\sqrt{6}$.

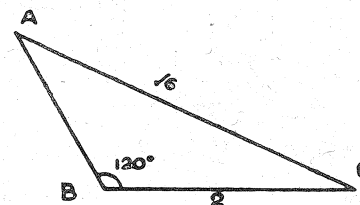


Fig. 13

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\therefore \sin A = \frac{a \sin B}{b} = \frac{2 \sin 120^\circ}{\sqrt{6}} = \frac{2\sqrt{3}}{2\sqrt{6}} = \frac{1}{\sqrt{2}}$$

$$\therefore A = 45^\circ \text{ or } 135^\circ.$$

Clearly the second value 135° of A is not admissible since $B + A = 120^\circ + 135^\circ > 180^\circ$, which is absurd.

$$\therefore \text{ the only admissible value of } A \text{ is } 45^\circ$$

$$\therefore C = 180^\circ - 120^\circ - 45^\circ = 15^\circ$$

$$\therefore c = \frac{a \sin C}{\sin A} = \frac{2 \sin 15^\circ}{\sin 45^\circ} = \frac{2(\sqrt{3}-1)}{2\sqrt{2} \cdot \frac{1}{\sqrt{2}}} = \sqrt{3}-1$$

$$\therefore \text{ solution is } A = 45^\circ, C = 15^\circ, c = \sqrt{3}-1.$$

INVERSE TRIGONOMETRICAL FUNCTIONS

11 Definition If an angle θ is given, we know all its trigonometric functions e.g., $\sin 30^\circ = \frac{1}{2}$, $\tan 45^\circ = 1$, and so on. But very often we are faced with the problem of finding the angle one of whose Trigonometric functions is given, e.g., if $\sin \theta = \frac{\sqrt{3}}{2}$ what is the value of θ ? or if $\tan \theta = \frac{1}{\sqrt{3}}$ what is the value of θ ? These equations have for answers values of θ that are not unique, in fact an infinite number of values of θ can be found which satisfy the equations above. These introduce us to 'the Inverse Trigonometric functions.'

If $\sin \theta = x$, we write $\theta = \sin^{-1}x$, and read it as 'sine inverse x ' or 'sine minus one x '. This stands for an angle whose sine is x and the angle may be one of many whose sine is x , e.g., $\sin \theta = \frac{1}{2}$ gives for θ values 30° , 150° , 390° , 510° , ... Similarly $\tan \theta = 1$ gives for θ the values 45° , 225° , 405° , 585° , ... The symbols $\sin^{-1}\frac{1}{2}$ and $\tan^{-1}1$ stand respectively for the values of θ mentioned above. Similarly, $\cos^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$ can all be defined.

In general if $y = f(x)$, then we say that $x = f^{-1}(y)$. This is called 'inverse notation'. It must be clearly understood that -1 here is not an algebraical index. In other words, $\sin^{-1}x$ is quite different from $(\sin x)^{-1}$. The former represents an angle while the latter represents a ratio $= \frac{1}{\sin x}$.

12 Principal Values We have already seen that $\sin^{-1}x$, $\cos^{-1}x$ etc. represent an infinite number of angles. The

numerically smallest of these is called the Principal value of the inverse function. Thus $\sin^{-1}\frac{1}{2} = 30^\circ$, $\sin^{-1}(-\frac{1}{2}) = -30^\circ$, $\tan^{-1}1 = 45^\circ$, $\tan^{-1}(-1) = -45^\circ$. In the case of the inverse cosine we are faced with a difficulty, e.g. $\cos^{-1}\frac{1}{2} = 60^\circ$ or -60° , $\cos^{-1}\frac{\sqrt{3}}{2} = 30^\circ$ or -30° , $\cos^{-1}(-\frac{1}{2}) = 120^\circ$ or -120° and so on. Thus $\cos^{-1}x$ gives two values equal in value but opposite in sign. Hence we modify the above definition and say that $\cos^{-1}x$ is the least positive angle whose cosine is x . Thus $\cos^{-1}\frac{1}{2} = 60^\circ$, $\cos^{-1}(-\frac{1}{2}) = 120^\circ$. With these definitions the Principal values of the inverse Trigonometric functions become unique. As further examples we can say that $\cot^{-1}1 = 45^\circ$, $\cot^{-1}(-1) = -45^\circ$, $\operatorname{cosec}^{-1}2 = 30^\circ$, $\operatorname{cosec}^{-1}(-2) = -30^\circ$, $\sec^{-1}2 = 60^\circ$, $\sec^{-1}(-2) = 120^\circ$.

Note: i) By defining the Principal values and introducing suitable restrictions the inverse relations are made one-one.

ii) In the case of the inverse sine and cosine, x must be numerically ≤ 1 for the angle to be real. The inverse secant and cosecant are real only if $|x| \geq 1$.

13 Graphs of Inverse Trigonometric functions

i) Let $y = \sin^{-1}x$ where $-1 \leq x \leq 1$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Plot the following sets of points and draw a smooth curve passing through them. The figure obtained represents the graph of $y = \sin^{-1}x$. (fig. 15)

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
y	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

ii) Let $y = \cos^{-1}x$ where $-1 \leq x \leq 1$, $0 \leq y \leq \pi$. Plot the following points and draw a smooth curve passing through them. We get the graph of $y = \cos^{-1}x$. (fig. 16)

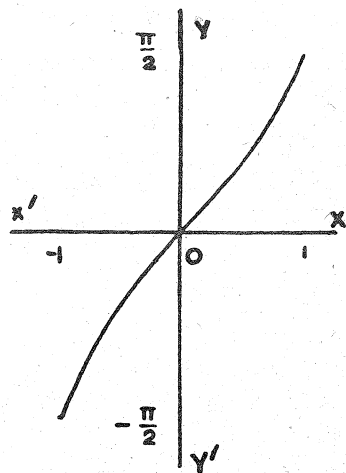


Fig. 15

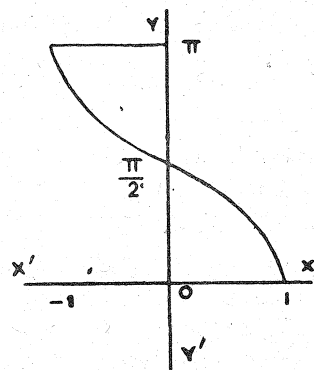


Fig. 16

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
y	π	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0

iii) Let $y = \tan^{-1}x$ where $-\infty < x < \infty$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Plot the following points and draw a smooth curve passing through them. We get the graph of $y = \tan^{-1}x$. (fig. 17)

x	$-\infty$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
y	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

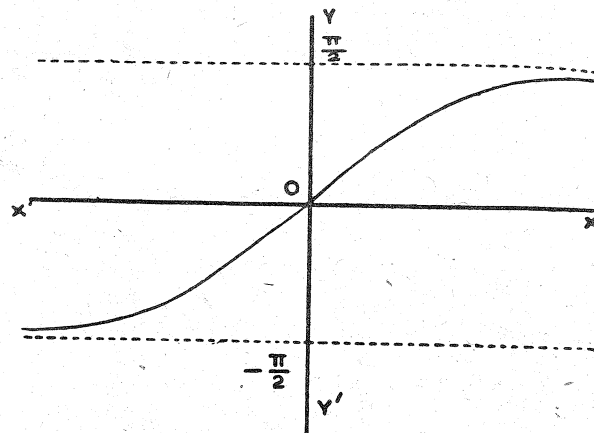


Fig. 17

Note: Compare the graphs of $y = \sin x$ and $y = \sin^{-1}x$. The latter is equivalent to $x = \sin y$. Hence if the point (x, y) lies on $y = \sin x$ then the point (y, x) lies on $x = \sin y$. The points (x, y) and (y, x) are symmetrically situated with respect to the line $y = x$. Hence the graphs of $y = \sin x$ and $y = \sin^{-1}x$ are images of each other in the line $y = x$.

Worked Examples

1) Find $\sin^{-1}(\sin 270^\circ)$.

$$\sin 270^\circ = -1,$$

$$\therefore \sin^{-1}(\sin 270^\circ) = \sin^{-1}(-1) = -90^\circ.$$

$\sin^{-1} 270$

2) Find $\cos^{-1}\left(\frac{1}{2}\tan\frac{\pi}{4}\right)$

$$\cos^{-1}\left(\frac{1}{2}\tan\frac{\pi}{4}\right) = \cos^{-1}\left(\frac{1}{2} \times 1\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ.$$

3) Find $\cos(\tan^{-1}\frac{3}{4})$.

Let $\tan^{-1}\left(\frac{3}{4}\right) = \alpha$. We know that α is an angle in the first quadrant.

Then $\tan \alpha = \frac{3}{4}$.

$$\therefore \cos(\tan^{-1}\frac{3}{4}) = \cos \alpha = \frac{1}{\sec \alpha} = \frac{1}{\sqrt{1 + \tan^2 \alpha}}$$

$$= \frac{1}{\sqrt{1 + \frac{9}{16}}} = \frac{4}{5}.$$

4) Find $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

$$\sin\frac{2\pi}{3} = \sin 120^\circ = \frac{\sqrt{3}}{2}$$

$$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

5) If $x \geq 0$, $y \geq 0$ and $xy \leq 1$, show that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

Let $\tan^{-1}x = \alpha$ and $\tan^{-1}y = \beta$.

Since $x \geq 0$ and $y \geq 0$ we know that α and β are angles in the first quadrant.

$\tan \alpha = x$ and $\tan \beta = y$.

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{x+y}{1-xy}.$$

$\tan(\alpha + \beta)$ is positive, by data,

$\therefore \alpha + \beta$ must be an angle in the first or the third quadrant.

But $\alpha + \beta < 180^\circ$, ($\because \alpha < 90^\circ$, $\beta < 90^\circ$),

$\therefore \alpha + \beta$ must be in the first quadrant

$$\therefore \alpha + \beta = \tan^{-1}\frac{x+y}{1-xy}.$$

6) Show that $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12} = \tan^{-1}\frac{17}{6}$

L.H.S. = $\tan^{-1}\frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}}$, from (5) above

$$= \tan^{-1}\frac{\frac{17}{12}}{\frac{1}{6}} = \tan^{-1}\frac{17}{6} = \text{R.H.S.}$$

7) Show that $\sin^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$.

$\sin^{-1}\frac{3}{5}$ is an angle in the first quadrant. By drawing a diagram it is easily seen that $\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$.

$$\therefore \text{L.H.S.} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{\frac{3}{4} + \frac{1}{4}}{1 - \frac{3}{4} \times \frac{1}{4}}$$

$$= \tan^{-1}\frac{\frac{2.5}{4}}{\frac{2.6}{4}} = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$$

8) Solve the equation $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$.

Let $\sin^{-1}x = \alpha$ so that $\sin \alpha = x$

Let $\sin^{-1}(1-x) = \beta$ so that $\sin \beta = 1-x$

We are given that $\alpha + \beta = \cos^{-1}x$

i.e., $\cos(\alpha + \beta) = x$

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta = x$$

$$\therefore \sqrt{1-x^2} \sqrt{1-(1-x)^2} - x(1-x) = x$$

$$\text{i.e., } \sqrt{1-x^2} \sqrt{2x-x^2} = 2x-x^2$$

$$\therefore \text{either } \sqrt{2x-x^2} = 0 \text{ or } \sqrt{1-x^2} = \sqrt{2x-x^2}$$

$$\text{i.e., } 2x-x^2 = 0 \text{ or } 1-x^2 = 2x-x^2$$

$$\text{i.e., } x(2-x) = 0 \text{ or } 2x = 1$$

$$\text{i.e., } x = 0, 2 \text{ or } \frac{1}{2}$$

$x = 2$ is impossible. $\therefore$ the solutions are $x = 0, \frac{1}{2}$.

Exercises

1 Find the values of:

i) $\tan^{-1}\sqrt{3}$ ii) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ iii) $\sin^{-1}[\sin(-600^\circ)]$

iv) $\cos^{-1}(\cos 350^\circ)$ v) $\cot^{-1}(-\sqrt{3})$ vi) $\cos^{-1}(0)$

vii) $\sec^{-1}(2)$ viii) $\sin^{-1}(\tan \pi)$ ix) $\sin^{-1}\left(\cos \frac{\pi}{9}\right)$

x) $\tan[\sin^{-1}(-\frac{3}{5})]$ xi) $\sin[\tan^{-1}(-1)]$
 xii) $\sin[\cos^{-1}(-\frac{3}{5})]$.

Answers: i) $\frac{\pi}{3}$ ii) $\frac{3\pi}{4}$ iii) 60° iv) 10° v) $\frac{\pi}{6}$
 vi) $\frac{\pi}{2}$ vii) $\frac{\pi}{3}$ viii) 0 ix) $\frac{7\pi}{18}$ x) $-\frac{3}{4}$
 xi) $-\frac{1}{\sqrt{2}}$ xii) $\frac{7}{25}$.

2 If $x \neq 0$, show that $\tan^{-1}x = \cot^{-1}(\frac{1}{x})$

3 If $x > 0$, show that $\tan^{-1}x + \cot^{-1}x = 90^\circ$

4 Prove that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

5 Show that $2 \tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2}$ if $0 < x < 1$

6 Show that $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{4} = \frac{\pi}{4}$

7 Show that $\tan^{-1}2 + \tan^{-1} \frac{1}{2} = \frac{\pi}{2}$

8 Show that $\sin^{-1} \frac{1}{\sqrt{5}} + \cot^{-1}(3) = \frac{\pi}{4}$

9 Draw the graph of $y = \sec^{-1}x$

10 Draw the graph of $y = \operatorname{cosec}^{-1}x$

11 Draw the graph of $y = \cot^{-1}x$

12 If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, prove that $x + y + z = xyz$.

Solve the following equations:

13 $\tan^{-1}(3x) + \tan^{-1}(2x) = \frac{\pi}{4}$

14 $\tan^{-1}x + \tan^{-1}(1-x) = \tan^{-1} \frac{9}{7}$

15 $\tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2}$

16 $\sin^{-1}(x) + \cos^{-1}(2x) = \frac{\pi}{6}$

17 $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$

Answers: 13) $\frac{1}{6}$ 14) $\frac{2}{3}, \frac{1}{3}$ 15) 0 16) $\frac{1}{2}$ 17) $\pm \frac{1}{\sqrt{2}}$

CHAPTER IV

GENERAL SOLUTION OF TRIGONOMETRIC EQUATIONS

14 To find the general expression for all angles which satisfy the equation $\sin \theta = k$.

Let $\sin \theta = k$, where $-1 \leq k \leq 1$

Then there is an angle α which satisfies the equation,

i.e. $\sin \theta = \sin \alpha$,

i.e. $\sin \theta - \sin \alpha = 0$,

i.e. $2 \sin \frac{\theta - \alpha}{2} \cos \frac{\theta + \alpha}{2} = 0$,

$\therefore$ either $\sin \frac{\theta - \alpha}{2} = 0$ or $\cos \frac{\theta + \alpha}{2} = 0$.

i) If $\sin \frac{\theta - \alpha}{2} = 0$ then $\frac{\theta - \alpha}{2} = p\pi$ where p is an integer or zero,

i.e. $\theta = 2p\pi + \alpha$

ii) If $\cos \frac{\theta + \alpha}{2} = 0$, then $\frac{\theta + \alpha}{2} = (2q + 1)\frac{\pi}{2}$, where q is any integer or zero.

i.e. $\theta = (2q + 1)\pi - \alpha$.

Combining both these results we get $\theta = n\pi + (-1)^n \alpha$, where n is any integer or zero.

Note: If α is in degrees then $\theta = n(180^\circ) + (-1)^n \alpha$.

15 To find the general expression for all angles satisfying the equation $\cos \theta = k$.

Let $\cos \theta = k$, where $-1 \leq k \leq 1$.

Then there is an angle α which satisfies the equation,

i.e. $\cos \theta = \cos \alpha$,

i.e. $\cos \theta - \cos \alpha = 0$,

i.e. $-2 \sin \frac{\theta + \alpha}{2} \sin \frac{\theta - \alpha}{2} = 0$.

i) If $\sin \frac{\theta+a}{2} = 0$ then $\frac{\theta+a}{2} = p\pi$ where p is an integer or 0,

$$\text{i.e. } \theta = 2p\pi - a.$$

ii) If $\sin \frac{\theta-a}{2} = 0$ then $\frac{\theta-a}{2} = q\pi$ where q is an integer or zero

$$\text{i.e. } \theta = 2q\pi + a.$$

Combining the two results we get $\theta = 2n\pi \pm a$, where n is an integer or zero.

Note: If a is in degrees then $\theta = n(360^\circ) \pm a$.

16 To find the general expression for all angles which satisfy the equation $\tan \theta = k$.

Let $\tan \theta = k$ where $-\infty < k < \infty$

There is an angle a which satisfies the equation,

$$\text{i.e. } \tan \theta = \tan a$$

$$\text{i.e. } \tan \theta - \tan a = 0$$

$$\text{i.e. } \frac{\sin \theta}{\cos \theta} - \frac{\sin a}{\cos a} = 0$$

$$\text{i.e. } \sin \theta \cos a - \cos \theta \sin a = 0$$

$$\text{i.e. } \sin(\theta - a) = 0$$

$$\text{i.e. } \theta - a = n\pi, \text{ where } n \text{ is an integer or zero}$$

$$\text{i.e. } \theta = n\pi + a.$$

Note: 1) If $\tan \theta = \tan a$ and a is in degrees then $\theta = n(180^\circ) + a$.

2) The equation $\operatorname{cosec} \theta = k$ is equivalent to $\sin \theta = \frac{1}{k}$. Hence the general solution for the equation is $\theta = n\pi + (-1)^na$ where a is one solution.

3) Similarly the general solutions for the equations $\sec \theta = k$ and $\cot \theta = k$ are respectively $\theta = 2n\pi \pm a$, $\theta = n\pi + a$, a being one solution.

Worked Examples

1 Solve the equation $\tan \theta = 1$

$$\tan \theta = 1 = \tan \frac{\pi}{4}.$$

One solution is evidently $\frac{\pi}{4}$.

$$\therefore \theta = n\pi + \frac{\pi}{4} \text{ where } n \text{ is any integer or zero.}$$

2 Solve the equation $\cos \theta = -\frac{1}{\sqrt{2}}$

$$\cos \theta = -\frac{1}{\sqrt{2}} = \cos \frac{3\pi}{4}.$$

One solution is $\frac{3\pi}{4}$.

$$\therefore \theta = 2n\pi \pm \frac{3\pi}{4} \text{ where } n \text{ is any integer or zero.}$$

3 Solve the equation $\sin \theta = \frac{1}{2}$

$$\sin \theta = \frac{1}{2} = \sin \frac{\pi}{6}$$

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}$, where n is any integer or zero.

4 Solve the equation $\sin 2\theta + \sin \theta = 0$

$$\sin 2\theta + \sin \theta = 0.$$

$$2 \sin \theta \cos \theta + \sin \theta = 0.$$

$$\sin \theta (2 \cos \theta + 1) = 0.$$

$\therefore$ either $\sin \theta = 0$ or $2 \cos \theta + 1 = 0$.

i) If $\sin \theta = 0$, then $\theta = n\pi$ where n is an integer or zero.

ii) If $2 \cos \theta + 1 = 0$, then $2 \cos \theta = -1$,

$$\text{i.e. } \cos \theta = -\frac{1}{2} = \cos \frac{2\pi}{3}$$

$$\therefore \theta = 2n\pi \pm \frac{2\pi}{3} \text{ where } n \text{ is an integer or zero.}$$

Combining the two results, the solution for the equation is $\theta = n\pi$ or $2n\pi \pm \frac{2\pi}{3}$ where n is an integer or zero.

5 Solve the equation $\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 0$

$$\sin \theta + \sin 7\theta + \sin 3\theta + \sin 5\theta = 0.$$

$$2 \sin 4\theta \cos 3\theta + 2 \sin 4\theta \cos \theta = 0.$$

$$2 \sin 4\theta (\cos 3\theta + \cos \theta) = 0.$$

$$2 \sin 4\theta (2 \cos 2\theta \cos \theta) = 0.$$

$$\text{Either } \sin 4\theta = 0 \text{ or } \cos 2\theta = 0 \text{ or } \cos \theta = 0$$

i) If $\sin 4\theta = 0$ then $4\theta = n\pi$, i.e. $\theta = \frac{n\pi}{4}$

ii) If $\cos 2\theta = 0$ then $2\theta = 2n\pi \pm \frac{\pi}{2}$,

$$\text{i.e. } \theta = n\pi \pm \frac{\pi}{4}$$

iii) If $\cos \theta = 0$ then $\theta = 2n\pi \pm \frac{\pi}{2}$

$\therefore$ the general solution for the given equation is

$$\theta = \frac{n\pi}{4}, n\pi \pm \frac{\pi}{4}, 2n\pi \pm \frac{\pi}{2} \text{ where } n \text{ is an integer or zero.}$$

6 Solve the equation $\cot \theta = \cot 3\theta$.

$$\cot 3\theta = \cot \theta,$$

$$\therefore 3\theta = n\pi + \theta, \text{ by Art. 16, note 3,}$$

$$\text{i.e. } 2\theta = n\pi,$$

$$\text{i.e. } \theta = \frac{n\pi}{2} \text{ where } n \text{ is an integer.}$$

7 Solve the equation $\sin 5\theta = \sin 3\theta$

$$\sin 5\theta = \sin 3\theta$$

$$\therefore 5\theta = n\pi + (-1)^n 3\theta, \text{ Art. 14,}$$

$$\therefore \theta [5 - (-1)^n 3] = n\pi$$

$$\text{i.e., } \theta = \frac{n\pi}{5 - (-1)^n 3}, \text{ where } n \text{ is a integer or zero.}$$

$$\text{If } n \text{ is odd, then } \theta = \frac{n\pi}{8}, \text{ if } n \text{ is even then } \theta = \frac{n\pi}{2}.$$

Exercises

Solve the following equations:

1 $2 \cos x + 1 = 0.$ 2 $\sqrt{2} \tan x = \sec x.$

3 $\tan 3x = 1.$ 4 $\tan 3x + \tan x = 0.$

5 $\cos 2x = \cos \left(x + \frac{\pi}{4}\right).$ 6 $\cos 2x = \sin \left(x + \frac{\pi}{4}\right).$

7 $\tan^2 x + 5 = 4 \sec x.$ 8 $\tan 5x = \cot 4x.$

9 $\sin^2 x = \frac{1}{2}.$ 10 $\cot^2 x = \frac{1}{2}.$

11 $\tan^2(2x) = \frac{1}{2}.$ 12 $\sin 2x = \sqrt{3} \cos x.$

13 $6 \cos^2 x + 5 \sin x - 7 = 0.$ 14 $\cot x - \tan x = 2.$

15 $\tan mx + \cot nx = 0.$ 16 $4 \sin^2 x - 8 \cos x + 1 = 0.$

17 $3(\sec^2 x + \tan^2 x) = 5.$ 18 $\sin x + \sin 2x + \sin 3x = 0.$

19 $\sin 2x - \cos 3x = \cos 2x - \sin 3x.$

20 $\cos 9x = \cos 5x - \cos x.$ 21 $\tan x + \sec x = \sqrt{3}.$

22 $\sqrt{3} \sec^2 x + 1 = (\sqrt{3} + 1) \tan x + \sqrt{3}.$

23 $\cos 5x + \sin 3x = 0.$ 24 $2 \sin^2 x + \sqrt{3} \cos x + 1 = 0.$

Answers: 1 $2n\pi \pm \frac{2\pi}{3}.$ 2 $n\pi + (-1)^n \frac{\pi}{4}.$

3 $\frac{1}{12}(4n+1)\pi.$ 4 $\frac{n\pi}{4}.$ 5 $2n\pi + \frac{\pi}{4}; \frac{1}{2}(2n\pi - \frac{\pi}{4}).$

6 $\frac{1}{2}(2n\pi + \frac{\pi}{4}); 2n\pi - \frac{\pi}{4}.$ 7 $2n\pi \pm \frac{\pi}{3}.$

8 $\frac{1}{9}(n\pi + \frac{\pi}{2}).$ 9 $n\pi \pm \frac{\pi}{6}.$ 10 $n\pi \pm \frac{\pi}{3}.$

11 $\frac{n\pi}{2} \pm \frac{\pi}{12}.$ 12 $2n\pi \pm \frac{\pi}{2}; n\pi + (-1)^n \frac{\pi}{3}.$

13 $n\pi + (-1)^n \frac{\pi}{6}; n(180^\circ) + (-1)^n 19^\circ 28'.$

14 $\frac{n\pi}{2} + \frac{\pi}{8}.$ 15 $\frac{1}{m-n} \left(k\pi + \frac{\pi}{2}\right).$ 16 $2n\pi \pm \frac{\pi}{3}.$

17 $n\pi \pm \frac{\pi}{6}.$ 18 $\frac{n\pi}{2}; 2n\pi \pm \frac{2\pi}{3}.$ 19 $4n\pi \pm \pi;$

$\frac{2}{3}(n\pi + \frac{\pi}{4}).$ 20 $\frac{1}{2}(2n\pi \pm \frac{\pi}{2}); \frac{1}{4}(2n\pi \pm \frac{\pi}{3}).$

$$21 \ 2n\pi + \frac{\pi}{6}. \quad 22 \ n\pi + \frac{\pi}{4}; n\pi + \frac{\pi}{6}. \quad 23 \ \frac{1}{2}(2n\pi + \frac{\pi}{2});$$

$$\frac{1}{8}(2n\pi - \frac{\pi}{2}). \quad 24 \ 2n\pi \pm \frac{5\pi}{6}.$$

17 To express $a \cos \theta + b \sin \theta$ in the form $R \cos(\theta - \alpha)$ and hence to solve the equation $a \cos \theta + b \sin \theta = c$.

Put $a = R \cos \alpha$ and $b = R \sin \alpha$

Then $a^2 + b^2 = R^2$, i.e., $R = \sqrt{a^2 + b^2}$

Also $\cos \alpha = \frac{a}{R}$, $\sin \alpha = \frac{b}{R}$, hence $\tan \alpha = \frac{b}{a}$

From any two of the three equations above, a value of α can be found.

$$\begin{aligned} \text{Then } a \cos \theta + b \sin \theta &= R \cos \alpha \cos \theta + R \sin \alpha \sin \theta \\ &= R (\cos \alpha \cos \theta + \sin \alpha \sin \theta) \\ &= R \cos(\theta - \alpha) \text{ where} \\ &\quad R = \sqrt{a^2 + b^2} \text{ and } \alpha \text{ is to be} \\ &\quad \text{found from the equations above.} \end{aligned}$$

The equation now becomes $R \cos(\theta - \alpha) = c$

$$\text{i.e., } \cos(\theta - \alpha) = \frac{c}{R}$$

Let β be an angle such that $\cos \beta = \frac{c}{R}$

Then $\cos(\theta - \alpha) = \cos \beta$.

$$\therefore \theta - \alpha = 2n\pi \pm \beta$$

$$\therefore \theta = \alpha + 2n\pi \pm \beta, \text{ where } n \text{ is any integer or zero.}$$

Note: i) If α and β are in degrees then $\theta = \alpha + n(360^\circ) \pm \beta$.

ii) The numerically greatest value of $a \cos \theta + b \sin \theta$ is seen to be $\sqrt{a^2 + b^2}$.

iii) The graph of $y = a \cos x + b \sin x$ can be easily drawn by first putting $a \cos x + b \sin x$ in the form $R \cos(x - \alpha)$.

Worked Example

Solve the equation $6 \sin \theta + 8 \cos \theta = 7$

Let $6 = R \sin \alpha$ and $8 = R \cos \alpha$

$$\text{Then } R^2 = 6^2 + 8^2 = 100$$

$$R = 10$$

$$\sin \alpha = \frac{6}{10} = \frac{3}{5}, \cos \alpha = \frac{8}{10} = \frac{4}{5}, \tan \alpha = \frac{3}{4}$$

α is evidently an angle in the first quadrant and its value is found to be $36^\circ 52'$, from Mathematical Tables.

We get on substitution,

$$R \sin \alpha \sin \theta + R \cos \alpha \cos \theta = 7$$

$$\text{i.e. } \cos(\theta - \alpha) = \frac{7}{R} = \frac{7}{10} = 0.7$$

Find the angle β whose cosine is 0.7. It is found to be $45^\circ 34.5'$

$$\therefore \cos(\theta - \alpha) = \cos \beta$$

$$\therefore \theta - \alpha = n(360^\circ) \pm \beta$$

$$\therefore \theta = n(360^\circ) \pm \beta + \alpha$$

$$= n(360^\circ) \pm 45^\circ 34.5' + 36^\circ 52'$$

$$= n(360^\circ) + 82^\circ 26.5' \text{ or } n(360^\circ) - 8^\circ 42.5'.$$

Exercises

Solve the following equations:

- 1 $\sin x + \cos x = \sqrt{2}$.
- 2 $\sin x + \sqrt{3} \cos x = 1$.
- 3 $\cos x + \sqrt{3} \sin x = 1$.
- 4 $\sqrt{3} \tan x = \sqrt{2} \sec x - 1$.
- 5 $6 \sin x + 8 \cos x = 5$.
- 6 $\sin x + 2 \cos x = 1$.
- 7 $\sqrt{3} \sin x - \cos x = \sqrt{2}$.
- 8 $\sin x - \cos x = \sqrt{2}$.
- 9 $3 \sin x + 4 \cos x = 5$.

$$\text{Answers: } 1 \ 2n\pi + \frac{\pi}{4}. \quad 2 \ 2n\pi + \frac{\pi}{2}; 2n\pi - \frac{\pi}{6}.$$

$$3 \ 2n\pi; 2n\pi + \frac{2\pi}{3}. \quad 4 \ 2n\pi + \frac{7\pi}{12}; 2n\pi + \frac{\pi}{12}$$

$$5 \ n(360^\circ) + 36^\circ 52' \pm 60^\circ. \quad 6 \ n(360^\circ) + 26^\circ 34' \pm 63^\circ 26'.$$

$$7 \ 2n\pi \pm \frac{3\pi}{4} - \frac{\pi}{3}. \quad 8 \ 2n\pi + \frac{3\pi}{4}.$$

$$9 \ n(360^\circ) + 36^\circ 52'.$$

ANALYTICAL GEOMETRY

CHAPTER I

STRAIGHT LINES

1 The angle between two straight lines

Let the equations of the straight lines AB and AC be $y = m_1x + c_1$ and $y = m_2x + c_2$. Let θ_1 and θ_2 be the angles made by AB and AC with OX and let θ be the angle between them (fig. 1).

We have $\tan \theta_1 = m_1$ and $\tan \theta_2 = m_2$.

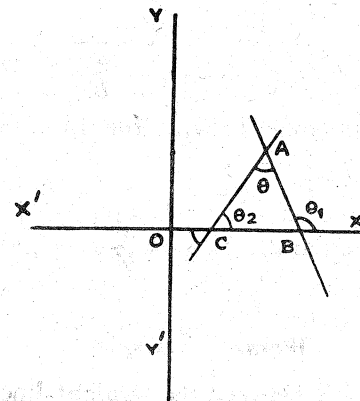


Fig. 1

From $\triangle ACB$ we get $\theta = \theta_1 - \theta_2$.

$$\begin{aligned}\tan \theta &= \tan (\theta_1 - \theta_2) \\ &= \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} \\ &= \frac{m_1 - m_2}{1 + m_1 m_2}.\end{aligned}$$

Note: θ is the acute or the obtuse angle between the lines according as $\frac{m_1 - m_2}{1 + m_1 m_2}$ is positive or negative.

Corollary 1. The lines are parallel if $\theta = 0$, i.e. $\tan \theta = 0$, i.e. $m_1 = m_2$

2. The lines are perpendicular if $\theta = \frac{\pi}{2}$, i.e. $\tan \theta = \infty$

i.e. $1 + m_1 m_2 = 0$ or $m_1 m_2 = -1$.

3. If the equations of the lines are given as $ax + by + c = 0$ and $a'x + b'y + c' = 0$, their slopes are $-\frac{a}{b}$ and $-\frac{a'}{b'}$

$$\therefore \tan \theta = \frac{-\frac{a}{b} + \frac{a'}{b'}}{1 + \left(-\frac{a}{b}\right)\left(-\frac{a'}{b'}\right)} = \frac{a'b - ab'}{aa' + bb'}$$

$\therefore$ the acute angle between the lines is given by

$$\tan \theta = \left| \frac{a'b - ab'}{aa' + bb'} \right|$$

The lines are parallel if $\frac{a}{b} = \frac{a'}{b'}$ and perpendicular if $aa' + bb' = 0$.

Worked Example

Find the angle between the straight lines $2x - 8y = 5$ and $12x - 2y = 7$.

The slopes of the lines are $\frac{1}{4}$ and 6.

The angle between the lines is given by

$$\tan \theta = \frac{6 - \frac{1}{4}}{1 + 6\left(\frac{1}{4}\right)} = \frac{23}{10}, \text{ i.e. } \theta = \tan^{-1}\left(\frac{23}{10}\right).$$

Exercises

1. Find the acute angle between the lines

i) $2x - y + 3 = 0$ and $x - 3y + 2 = 0$. **Ans.** 45°

ii) $3x - 2y + 1 = 0$ and $2x + y + 5 = 0$.

Ans. $\tan^{-1} \frac{7}{4}$

iii) $x + y\sqrt{3} + 2 = 0$ and $\sqrt{3}x - y + 1 = 0$.

Ans. 90°

2. Show that the straight lines $3x + y + 2 = 0$, $3x + 4y + 5 = 0$ and $24x - 7y + 10 = 0$ form an isosceles triangle.

3. Find the equations of the lines that pass through the origin and make angles of 45° with $y = 3x + 2$.

Ans. $y = \frac{x}{2}$; $y = -2x$

4. Find the equations of the lines that pass through (1, 2) and make angles of 60° with $\sqrt{3}x + y = 2$.

Ans. $y = 2$; $y - 2 = \sqrt{3}(x - 1)$

5. Find the equations of the lines passing through the point $(-1, 1)$ and making angles of $\tan^{-1} \frac{1}{3}$ with the line $x + 2y = 7$.

Ans. $x + y = 0$,
 $x + 7y = 6$

2. Sign of $ax + by + c$

Let $ax + by + c = 0$ be the equation of the line AB . Let P and Q be the points (x_1, y_1) and (x_2, y_2) . Let AB cut PQ or PQ produced at R . Let R divide PQ in the ratio $k : 1$. Then the coordinates of R are $\left(\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1}\right)$.

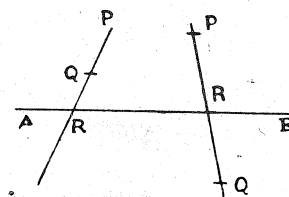


Fig. 2

Since R lies on the line $ax + by + c = 0$,

$$a\left(\frac{kx_2 + x_1}{k + 1}\right) + b\left(\frac{ky_2 + y_1}{k + 1}\right) + c = 0,$$

$$\text{i.e. } a(kx_2 + x_1) + b(ky_2 + y_1) + c(k + 1) = 0$$

$$\text{i.e. } k = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$$

If P and Q are on the same side of AB , PQ is divided externally at R , $\therefore k$ is $-ve$.

$\therefore ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign.

If P and Q are on opposite sides of AB , PQ is divided internally at R , $\therefore k$ is $+ve$.

$\therefore ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite signs.

Hence we conclude that $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign or opposite signs according as P and Q are on the same side or opposite sides of AB .

Note: In particular if Q is taken as the origin, i.e. the point $(0, 0)$ then $ax_2 + by_2 + c \equiv c$; hence (x_1, y_1) and $(0, 0)$ are on the same side or on opposite sides of $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and c are of the same sign or opposite signs.

3 Length of the perpendicular from (x_1, y_1) to the straight line $x \cos a + y \sin a = p$

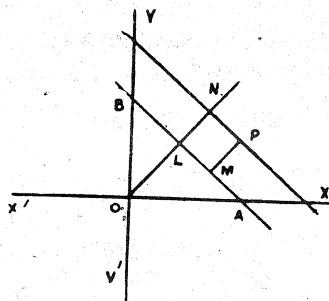


Fig. 3

Let AB be the given line and P be the given point (x_1, y_1) .

Draw PM perpendicular to AB .

Draw the line through P parallel to AB .

Draw OL perpendicular to AB to cut the parallel through P in N .

We know that $OL = p$ and $\angle XOL = a$.

Let $ON = p'$

Then the equation to NP is $x \cos a + y \sin a = p'$

Since this passes through $P(x_1, y_1)$, we have

$$x_1 \cos a + y_1 \sin a = p'$$

Now $MP = LN$

$$= ON - OL$$

$$= p' - p$$

$$= x_1 \cos a + y_1 \sin a - p$$

$\therefore$ the required perpendicular distance =

$$x_1 \cos a + y_1 \sin a - p.$$

Note: 1. If P and O are on the same side of AB then the required perpendicular distance $= p - p' = p - x_1 \cos a - y_1 \sin a$. (fig. 4)

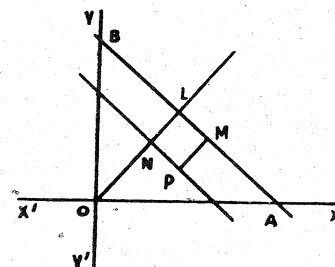


Fig. 4

2. The length of the perpendicular distance $= |x_1 \cos a + y_1 \sin a - p|$.

4 Length of the perpendicular distance of (x_1, y_1) from $ax + by + c = 0$

Let us assume that c is negative; if c is positive the equation can be rewritten with the constant term negative.

The equation of the line can be put in the form

$$\frac{a}{\sqrt{a^2 + b^2}} x + \frac{b}{\sqrt{a^2 + b^2}} y + \frac{c}{\sqrt{a^2 + b^2}} = 0$$

$$\text{Let } \frac{a}{\sqrt{a^2 + b^2}} = \cos a, \text{ then } \frac{b}{\sqrt{a^2 + b^2}} = \sin a,$$

for we know that $\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$

$\therefore$ the given line is of the form

$$x \cos a + y \sin a + \frac{c}{\sqrt{a^2+b^2}} = 0.$$

$\therefore$ the distance of (x_1, y_1) from the line

$$\begin{aligned} &= \left| x_1 \cos a + y_1 \sin a + \frac{c}{\sqrt{a^2+b^2}} \right|, \text{ by Art. 3,} \\ &= \left| x_1 \frac{a}{\sqrt{a^2+b^2}} + y_1 \frac{b}{\sqrt{a^2+b^2}} + \frac{c}{\sqrt{a^2+b^2}} \right| \\ &= \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|. \end{aligned}$$

Note: If (x_1, y_1) and (x_2, y_2) are on the same side of the line, then the perpendiculars from them to the line are of the same sign; if the points are on opposite sides of the line then the perpendiculars are of opposite signs (Art. 2).

Worked Examples

- 1 Find the perpendicular distance of the line $3x + 4y - 2 = 0$ from the point $(3, 7)$.

$$\text{The perpendicular distance} = \frac{3(3) + 4(7) - 2}{\sqrt{3^2 + 4^2}} = \frac{35}{5} = 7.$$

2. Find the distance of the line $x - 3y + 12 = 0$ from the point $(1, -2)$.

$$\text{Distance required} = \frac{1 - 3(-2) + 12}{\sqrt{1 + 9}} = \frac{19}{\sqrt{10}} = \frac{19}{10} \sqrt{10}.$$

- 3 Find the distance between the parallel lines $4x + 3y = 7$ and $4x + 3y = 17$.

Putting $y=0$ in the equation $4x + 3y = 7$, we get, $x = \frac{7}{4}$.

One point on the line is $(\frac{7}{4}, 0)$.

The required distance between the parallel lines is the distance of $(\frac{7}{4}, 0)$ from the other line $4x + 3y = 17$.

$$\text{Required distance} = \left| \frac{4(\frac{7}{4}) - 17}{\sqrt{4^2 + 3^2}} \right| = \frac{10}{5} = 2.$$

Exercises

1 Show that the points $(1, 1)$ and $(4, 3)$ are on opposite sides of the line $4x - 3y - 6 = 0$.

2 Show that the origin and $(2, -1)$ are on the same side of the line $2x + 5y - 1 = 0$.

3. Find the length of the perpendicular drawn from the point

i) $(4, 5)$ on the line $3x - 4y = 7$ Ans. 3

ii) $(3, -2)$ on the line $5x - 12y = 9$ Ans. $\frac{39}{13}$

iii) $(3, 1)$ on the line joining the points $(4, 1)$ and $(6, 4)$

$$\text{Ans. } \frac{3\sqrt{13}}{13}$$

iv) (b, a) on the line $\frac{x}{a} - \frac{y}{b} = 1$. Ans. $\left| \frac{b^2 - a^2 - ab}{\sqrt{a^2 + b^2}} \right|$

4 Show that the point $(1, 1)$ is equidistant from the lines $3x - 4y + 21 = 0$ and $15x + 8y + 45 = 0$.

5 Find the distance between the parallel lines

i) $4x - 3y = 6$ and $4x - 3y = 2$ Ans. $\frac{4}{5}$

ii) $2x + y = 11$ and $2x + y = 14$ Ans. $\frac{3}{5}\sqrt{5}$

6 Find the equation of the locus of a point which moves such that its distance from the point $(a, 0)$ is equal to its distance from the line $x + a = 0$. Ans. $y^2 = 4ax$

7 Find the equation of the locus of a point which moves so that its distance from the point $(-1, 1)$ is five times its distance from the line $3x + 4y = 2$.

$$\text{Ans. } 8x^2 + 24xy + 15y^2 - 14x - 14y + 2 = 0$$

8 Show that the distances of any point on the line $3x = 11y$ from the lines $12x - 5y + 52 = 0$ and $3x + 4y + 20 = 0$ are equal

9 The algebraic sum of the perpendiculars from two points P and Q on a line is zero. Prove that the line passes through the midpoint of PQ .

10 Show that the line $5x + 12y = 4$ is a tangent to the circle whose centre is $(3, -2)$ and radius is 1.

5 The equation of the bisectors of the angles between two straight lines

Let the given lines be $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$.
Let the lines intersect at Q .

Draw the bisectors of the angles BQD and BQC .

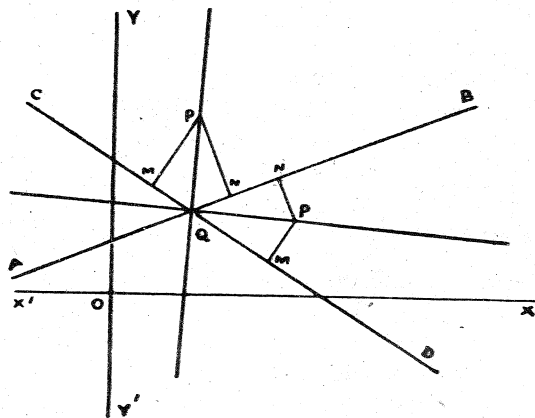


Fig. 5

If P be any point (x, y) on either bisector, we know that the distances of P from AB and CD are equal.

Let P be any point on the bisector of angle BQC . Draw PM and PN perpendicular to CD and AB . Then $PM = PN$ in magnitude, (fig. 5).

Let us assume that the given equations have the constants c_1 and c_2 both **negative**. This can be done without any loss of generality.

P and O are on opposite sides of CD .

$\therefore$ the length of the perpendicular $PM = \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}$

P and O are on opposite sides of AB also.

$\therefore$ the length of the perpendicular $PN = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

$\therefore$ the equation of the bisector of angle BQC is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

If P is any point on the other bisector, arguing as before, we find that P and O lie on the same side of AB but on opposite sides of CD . Hence if we equate the perpendicular distances PM and PN we get

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = - \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

This is the equation of the bisector of angle BQD .

Note: The $+$ sign gives the equation of the bisector of the angle in which the origin lies and the $-$ sign gives the equation of the bisector of the angle in which the origin does not lie; provided that the constants c_1 and c_2 are both of the same sign.

Worked Examples

1 Find the equations of the bisectors of the angles between the lines $3x + 4y + 1 = 0$ and $5x - 12y - 4 = 0$.

The equations of the lines can be written as $-3x - 4y - 1 = 0$ and $5x - 12y - 4 = 0$. The constant terms have been made negative in both the equations.

The equation of the bisector of the angle which contains the origin is given by

$$\frac{-3x - 4y - 1}{5} = + \frac{5x - 12y - 4}{13}$$

i.e. $13(-3x - 4y - 1) = 5(5x - 12y - 4)$

i.e. $64x - 8y - 7 = 0$.

The equation of the other bisector is

$$\frac{-3x - 4y - 1}{5} = -\frac{5x - 12y - 4}{13}$$

i.e. $13(-3x - 4y - 1) = -5(5x - 12y - 4)$

i.e. $14x + 112y + 33 = 0$.

2 Find the equations of the bisectors of the angles between the lines $x + y - 3 = 0$ and $7x - y + 5 = 0$ and state which of them bisects the acute angle between the lines.

The lines are $x + y - 3 = 0$ and $-7x + y - 5 = 0$.

The equations of the bisectors are

$$\frac{x+y-3}{\sqrt{2}} = \pm \frac{-7x+y-5}{5\sqrt{2}}, \text{ the } + \text{ sign giving the bisector of the angle which contains the origin.}$$

When simplified these reduce to $6x + 2y - 5 = 0$ and $x - 3y + 10 = 0$.

The slope of the line $x - 3y + 10 = 0$ is $\frac{1}{3}$

The slope of the line $x + y - 3 = 0$ is -1

The angle between these two lines is $\tan^{-1} \frac{\frac{1}{3} - (-1)}{1 + \frac{1}{3}(-1)}$
 $= \tan^{-1}(2) > 45^\circ$.

This is therefore the bisector of the obtuse angle between the lines. Therefore the required bisector is

$$6x + 2y - 5 = 0.$$

Exercises

1 Find the equations of the bisectors of the angles between

a) $4x + 3y - 1 = 0$ and $3x - 4y + 5 = 0$

b) $12x - 5y + 7 = 0$ and $4x - 3y + 8 = 0$

c) $3x - y + 5 = 0$ and $x + 3y - 4 = 0$

d) $5x + 12y - 1 = 0$ and $7x - 24y + 2 = 0$.

Ans. a) $x + 7y - 6 = 0$; $7x - y + 4 = 0$

b) $8x + 14y - 69 = 0$, $112x - 64y + 139 = 0$

c) $2x - 4y + 9 = 0$, $4x + 2y + 1 = 0$

d) $2x + 36y - 3 = 0$, $216x - 12y + 1 = 0$

2 Find the equation of the bisector of the angle between $4x + y - 7 = 0$ and $x - 4y + 3 = 0$ which contains the origin.

Ans. $5x - 3y - 4 = 0$

3 Find the equation of the bisector of the angle between $12x - 5y + 3 = 0$ and $3x + 4y - 7 = 0$ which does not contain the origin.

Ans. $21x - 77y + 106 = 0$

4 Find the equations of the bisectors of the angles between $x\sqrt{3} - y = 2$ and $x - y\sqrt{3} = 4$ and distinguish between them.

Ans. $x + y + \sqrt{3} + 1 = 0$

$x - y = 3(\sqrt{3} - 1)$

5 Find the equation of the bisector of the acute angle between the lines $2x = y + 4$ and $2y = x + 10$.

Ans. $x - y + 2 = 0$

6 The point of intersection of two straight lines

Let the two straight lines be $ax + by + c = 0 \dots (1)$

and $a'x + b'y + c' = 0 \dots (2)$

The coordinates of the point of intersection of the two straight lines satisfy the equations (1) and (2) simultaneously, because the point lies on both the straight lines. Therefore the coordinates of the point of intersection are obtained by solving the equations (1) and (2) simultaneously.

By the method of cross-multiplication we have,

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{1}{ab' - a'b}$$

$$\therefore x = \frac{bc' - b'c}{ab' - a'b} \text{ and } y = \frac{ca' - c'a}{ab' - a'b}$$

$$\therefore \text{the point of intersection is } \left(\frac{bc' - b'c}{ab' - a'b}, \frac{ca' - c'a}{ab' - a'b} \right).$$

Corollary:—

If the straight lines are $\parallel$, their point of intersection will be at infinity. $\therefore ab' - a'b = 0$ i.e., $\frac{a}{a'} = \frac{b}{b'}$.

7 The equation of a straight line through the point of intersection of two straight lines.

Let the straight lines be $L \equiv ax + by + c = 0 \dots (1)$

and $L' \equiv a'x + b'y + c' = 0 \dots (2)$

Consider the equation

$$ax + by + c + k(a'x + b'y + c') = 0 \dots (3)$$

i.e., $L + kL' = 0$ where k is any constant. Equation

(3) is linear in x and y and therefore it represents a straight line. If (x_1, y_1) is the point of intersection of the straight lines (1) and (2), we have $ax_1 + by_1 + c = 0$

$$\text{and } a'x_1 + b'y_1 + c' = 0,$$

$$\text{so that } ax_1 + by_1 + c + k(a'x_1 + b'y_1 + c') = 0$$

$\therefore$ the straight line (3) passes through (x_1, y_1) i.e., the point of intersection of the straight lines (1) and (2).

$\therefore$ any straight line passing through the point of intersection of (1) and (2) is given by the equation $ax + by + c + k(a'x + b'y + c') = 0$, where k is a constant.

Note: This form of the equation is useful in problems where the actual point of intersection of two straight lines is not required.

Worked Examples

1 Find (i) the point of intersection of the lines $x + 7y - 23 = 0$ and $5x + 2y - 16 = 0$; (ii) the equation of the line joining this point with $(1, 0)$; (iii) the equation of the line perpendicular to the line in (ii) and passing through the origin.

(i) The point of intersection of the lines

$$x + 7y - 23 = 0 \dots (1)$$

$$\text{and } 5x + 2y - 16 = 0 \dots (2)$$

is obtained by solving the equations (1) and (2)

$$\therefore \begin{array}{l} 5x + 2y = 16 \\ \text{and } 5x + 35y = 115 \end{array}$$

$$\therefore -33y = -99$$

$$\therefore y = 3$$

$$\therefore x = 2$$

$\therefore$ the point of intersection is $(2, 3)$

(ii) the slope of the line through $(2, 3)$ and $(1, 0)$
 $= \frac{3-0}{2-1} = 3.$

$\therefore$ the equation of the line through $(1, 0)$ and $(2, 3)$ is given by $y - 0 = 3(x - 1)$ i.e., $3x - y - 3 = 0.$

(iii) The equation of the line $\perp$ to the line in (2) is of the form $x + 3y = a$. Since it passes through the origin $a = 0$

$\therefore$ the required equation is $x + 3y = 0.$

2 Find the equation of the straight line that passes through the point of intersection of the straight lines $3x - 4y = 7$ and $12x + 5y = 15$ and is perpendicular to the straight line $2x - 3y + 5 = 0.$

The straight lines are

$$3x - 4y = 7 \dots (1)$$

$$12x + 5y = 15 \dots (2)$$

Any straight line passing through the point of intersection of (1) and (2) is given by the equation

$$3x - 4y - 7 + k(12x + 5y - 15) = 0.$$

$$\text{i.e., } (3 + 12k)x + (5k - 4)y - 7 - 15k = 0, \dots (3)$$

If this line is $\perp$ to $2x - 3y + 5 = 0$,

$$\text{then } (3 + 12k) \cdot 2 + (5k - 4) \cdot (-3) = 0$$

$$\text{i.e. } 9k + 18 = 0, \quad \text{i.e. } k = -2.$$

$\therefore$ the required straight line is

$$(3 - 24)x + (-10 - 4)y - 7 + 30 = 0$$

$$\text{i.e. } -21x - 14y + 23 = 0$$

$$\text{i.e. } 21x + 14y - 23 = 0.$$

Note: This question can also be solved by finding the point of intersection of (1) and (2),

3 Find the ortho-centre of the triangle formed by $(5, -2)$, $(-1, 2)$ and $(1, 4)$.

Let the vertices be A, B, C in order and let AD, BE be two altitudes intersecting at the ortho-centre H .

$$\text{The slope of } BC = \frac{4 - 2}{1 - (-1)} = 1$$

$$\therefore \text{the slope of } AD (\perp \text{ to } BC) = -1$$

$$\therefore \text{the equation to } AD \text{ is } y + 2 = -(x - 5) \\ \text{i.e., } x + y - 3 = 0 \dots (1)$$

$$\text{The slope of } CA = \frac{4 - (-2)}{1 - 5} = -\frac{3}{2}$$

$$\therefore \text{the slope of } BE (\perp \text{ to } CA) = \frac{2}{3}$$

$$\therefore \text{the equation to } BE \text{ is } y - 2 = \frac{2}{3}(x + 1) \\ \text{i.e., } 3y - 6 = 2x + 2 \\ \text{i.e., } 2x - 3y + 8 = 0 \dots (2)$$

The ortho-centre H is the point of intersection of (1) and (2).

$$\begin{array}{rcl} \therefore 2x - 3y & = & -8 \\ 2x + 2y & = & 6 \\ \hline \therefore -5y & = & -14 \end{array} \quad \therefore y = \frac{14}{5}$$

$$\text{and } x = \frac{1}{5}$$

$$\therefore H \text{ is the point } \left(\frac{1}{5}, \frac{14}{5}\right).$$

4 Show that the lines $3x + 4y = 13$, $2x - 7y + 1 = 0$ and $5x - y = 14$ are concurrent.

$$\text{The given lines are } 3x + 4y = 13 \dots (1)$$

$$2x - 7y = -1 \dots (2)$$

$$5x - y = 14 \dots (3)$$

The point of intersection of (1) and (3) is got by solving the two equations.

$$\therefore 3x + 4y = 13$$

$$20x - 4y = 56$$

$$\hline 23x = 69$$

$$\therefore x = 3$$

$$\text{and } y = 1$$

$$\therefore \text{the point of intersection is } (3, 1)$$

The coordinates of this point satisfy equation (2), as can be verified by substitution.

$\therefore$ the three straight lines are concurrent at the point $(3, 1)$.

Exercises

1 Find the equation of the line through the point of intersection of the lines $x + 3y = 1$ and $x - 2y + 4 = 0$ and $\parallel$ to the straight line $3x + 4y = 0$. Ans. $3x + 4y + 2 = 0$

2 Find the coordinates of the point of intersection of the straight lines $4x + 3y = 10$ and $3x + 5y = 13$. Obtain the equation of the straight line passing through this point and perpendicular to the former.

$$\text{Ans. } (1, 2); 3x - 4y + 5 = 0$$

3 Given the lines $y = x$ and $y = 2 - x$ find (i) their point of intersection (ii) the equation of the line perpendicular to $y = 4x + 5$, through the point of intersection and (iii) the area included by the lines $y = x$, $y = 2 - x$ and $y = 0$.

$$\text{Ans. (i) } (1, 1), \text{ (ii) } x + 4y = 5, \text{ (iii) } 1$$

4 Find the equation of the st. line passing through the point of intersection of $2x + 3y = 1$ and $3x - 4y = 6$ and (i) $\parallel$ to $5x - y = 0$ (ii) $\perp$ to $x + 2y = 3$.

$$\text{Ans. (i) } 5x - y = 7, \text{ (ii) } 34x - 17y = 53$$

5 Find the equation of the straight line which passes through the intersection of $x + 2y + 5 = 0$ and $y = x + 7$ and is $\parallel$ to $5x - 2y + 1 = 0$. Ans. $5x - 2y + 33 = 0$

6 Find the equation of the straight line joining the point $(5, -2)$ to the intersection of the lines $2x - y + 5 = 0$ and $x + y + 1 = 0$. Ans. $3x + 7y = 1$

7 Find the coordinates of the ortho-centre of the triangle whose vertices are $(1, -2)$, $(3, 1)$ and $(-2, 3)$.

$$\text{Ans. } \left(\frac{1}{3}, \frac{1}{3}\right)$$

8 Find the coordinates of the ortho-centre of the triangle formed by the points $(-3, -7)$, $(13, 2)$ and $(-5, 6)$.

$$\text{Ans. } (-3, 2)$$

- 9 Find the circum-centre of the triangle formed by the points $(-2, 3)$, $(2, -1)$ and $(4, 0)$.
Ans. $(\frac{3}{2}, \frac{5}{2})$
- ✓ 10 Find the circum-centre of the triangle whose vertices are $(5, 3)$, $(1, 5)$ and $(3, -1)$.
Ans. $(2, 2)$
- ✓ 11 The vertices of a quadrilateral taken in order are $(-1, -2)$, $(2, 1)$, $(3, 4)$ and $(1, 2)$. Find the equations of the diagonals and their point of intersection.
Ans. $3x - 2y = 1$, $x + y = 3$, $(\frac{7}{5}, \frac{8}{5})$
- ✓ 12 Show that the straight lines $2x - 3y + 12 = 0$, $2x - y + 8 = 0$ and $4x + 5y + 2 = 0$ are concurrent. Find the point of concurrence.
Ans. $(-3, 2)$
- ✓ 13 Find the value of a so that the lines $x - 6y + a = 0$, $2x + 3y + 4 = 0$ and $x + 4y + 1 = 0$ may be concurrent.
Ans. 5
- ✓ 14 Find the value of k such that $2x + ky + 2 = 0$ may pass through the point of intersection of $4x - 3y = 7$ and $6x + 5y = 1$.
Ans. 4
- ✓ 15 The line $x - 3y + 7 = 0$ cuts two other lines $2x - y = 1$ and $3x + 2y = 23$ at points A and B . Find the length of AB . Find also the line through $(3, 4) \parallel$ to AB .
Ans. $\sqrt{10}$; $x - 3y + 9 = 0$
- ✓ 16 Find the equation of the line joining the point of intersection of the lines $3x + 2y + 1 = 0$, $x + y = 3$ to the point of intersection of the lines $y - x = 1$, $y + 2x + 2 = 0$.
Ans. $5x + 3y + 5 = 0$
- ✓ 17 Find the equation of the line through the point of intersection of the lines $2x - 5y + 1 = 0$ and $3x + 2y = 8$ and cutting off equal intercepts from the axes.
Ans. $x + y = 3$, $x - y = 1$
- ✓ 18 Show that $(2 + 5k)x - 3(1 + 2k)y + 2 - k = 0$ represents, for different values of k , a concurrent system of straight lines. Find the point of concurrence. Find the lines through this point, parallel to the axes.
Ans. $(5, 4)$, $x = 5$, $y = 4$

8 Condition for concurrency of three lines

Let the equations of the three lines be $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$. If the three lines are concurrent then the point of intersection of any two of them lies on the third. The point of intersection of lines (2) and (3) is

$$\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}, \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2} \right)$$

If this point lies on line (1) then

$$a_1 \left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2} \right) + b_1 \left(\frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2} \right) + c_1 = 0$$

or $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$.

This can also be put in the determinant form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

9 Another condition for concurrency of three lines

If constants l, m, n (not all zero) can be found such that $l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) + n(a_3x + b_3y + c_3) = 0$ identically, then the three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ are concurrent.

Let $l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) + n(a_3x + b_3y + c_3) \equiv 0 \dots (A)$

Let $l \neq 0$. Then if lines 2 and 3 meet at (x_1, y_1) we have $a_2x_1 + b_2y_1 + c_2 = 0$ and $a_3x_1 + b_3y_1 + c_3 = 0$. Since (A) above is an identity, it must be true for all values of x and y .

In particular it must be true for $x = x_1$ and $y = y_1$

$$\therefore l(a_1x_1 + b_1y_1 + c_1) + m(a_2x_1 + b_2y_1 + c_2) + n(a_3x_1 + b_3y_1 + c_3) = 0$$

$$+ n(a_3x_1 + b_3y_1 + c_3) = 0$$

$$\text{i.e., } l(a_1x_1 + b_1y_1 + c_1) = 0$$

$$\text{But } l \neq 0, \text{ hence } a_1x_1 + b_1y_1 + c_1 = 0$$

i.e. the line $a_1x + b_1y + c_1 = 0$ passes through (x_1, y_1)

i.e. the three lines are concurrent.

Worked Example

Show that the lines given by the equations

$2x - 3y - 7 = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$, are concurrent.

We have identically, $2x - 3y - 7 + 2(3x - 4y - 13) - (8x - 11y - 33) = 0$

Hence by the above test for concurrency, the three lines are concurrent.

Exercises

1 Prove that the lines $(a-b)x + (b-c)y = c-a$, $(b-c)x + (c-a)y = a-b$ and $(c-a)x + (a-b)y = b-c$ are concurrent.

2 Prove that the internal bisectors of the angles of a triangle are concurrent. [Hint. Take the origin within the triangle]

3 Prove that the lines $4x + 7y = 9$, $5x + 15 = 8y$, $y = 9x + 6$ are concurrent. Find the point of concurrency.

Ans. $\left(-\frac{33}{67}, \frac{105}{67}\right)$

4 Show that the lines $px + qy + r = 0$, $qx + ry + p = 0$, and $rx + py + q = 0$ are concurrent, if $p + q + r = 0$.

5 Find the condition that the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ may be concurrent.

Ans. $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$

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CHAPTER II

THE CIRCLE

10 The equation of a circle whose centre is the origin and whose radius is a .

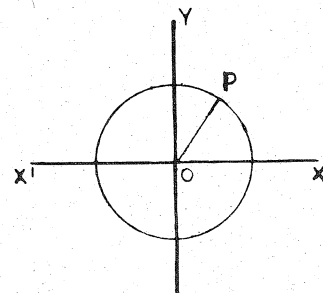


Fig. 6

Let $P(x, y)$ be any point on the circle whose centre is O and radius $= a$.

Then $OP = a$

$$\therefore OP^2 = a^2$$

$$\text{i.e., } x^2 + y^2 = a^2 \dots (1)$$

Equation (1) is satisfied by every point on the circle.

$\therefore$ the equation of the circle is $x^2 + y^2 = a^2$.

11 The equation of a circle whose centre is (h, k) and whose radius is a .

Let $C(h, k)$ be the centre of the circle and let $P(x, y)$ be any point on the circle.

Then $CP = a$

$$\therefore CP^2 = a^2$$

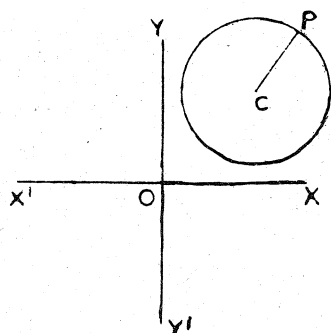


Fig. 7

$$\text{i.e., } (x-h)^2 + (y-k)^2 = a^2 \dots (1)$$

Equation (1) is satisfied by every point on the circle.

$\therefore$ the equation of the circle is $(x-h)^2 + (y-k)^2 = a^2$.

$$\text{i.e., } x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0.$$

12 Standard equation of a circle

An equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle. For, the equation can be put in the form $(x+g)^2 + (y+f)^2 = (\sqrt{g^2 + f^2 - c})^2$ which shows that the distance of any point (x, y) on the locus from the fixed point $(-g, -f)$ is always constant and equal to $\sqrt{g^2 + f^2 - c}$. $\therefore$ the locus of (x, y) is a circle whose centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Note: It should be noted that in the standard equation of a circle,

- (1) the coefficients of x^2 and y^2 are equal to unity and
- (2) the coefficient of $xy = 0$
- (3) the centre is given by $(-g, -f)$ i.e.,
the x -coordinate of the centre $= -\frac{1}{2}$ coefficient of x
the y -coordinate of the centre $= -\frac{1}{2}$ coefficient of y .
- (4) the radius is $\sqrt{g^2 + f^2 - c}$.
- (5) if the equation is given as

$ax^2 + ay^2 + 2gx + 2fy + c = 0$, divide by a to get the equation of the circle in the standard form.

Worked Examples

- 1 Find the equation of the circle whose centre is $(-1, 3)$ and radius is 5.

The equation of the circle is $(x+1)^2 + (y-3)^2 = 5^2$
i.e., $x^2 + y^2 + 2x - 6y - 15 = 0$.

- 2 Find the centre and radius of the circle $x^2 + y^2 - 6x + 8y - 11 = 0$.

The centre is $(3, -4)$ and radius $= \sqrt{3^2 + 4^2 + 11} = 6$

- 3 Find the equation of the circle passing through the points $(2, 3)$, $(4, -3)$, $(5, -2)$.

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Since it passes through the given points, we have

$$4 + 9 + 4g + 6f + c = 0$$

$$16 + 9 + 8g - 6f + c = 0 \text{ and}$$

$$25 + 4 + 10g - 4f + c = 0$$

$$\text{i.e., } 4g + 6f + c = -13 \dots (1)$$

$$8g - 6f + c = -25 \dots (2)$$

$$10g - 4f + c = -29 \dots (3)$$

Eliminating c between (1) and (2), (2) and (3) we get,

$$-4g + 12f = 12 \text{ and } -2g - 2f = 4$$

$$\text{i.e., } -g + 3f = 3 \dots (4)$$

$$\text{and } -g - f = 2 \dots (5)$$

From (4) and (5) we get $g = -\frac{3}{4}$, $f = \frac{1}{4}$

$\therefore$ from (1) we get $c = -\frac{11}{4}$.

$\therefore$ the equation of the circle is

$$x^2 + y^2 - \frac{3}{2}x + \frac{1}{2}y - \frac{11}{4} = 0$$

$$\text{or } 2(x^2 + y^2) - 3x + y - 11 = 0.$$

- 4 Find the equation of the circle which passes through the points $(4, -10)$ and $(-8, 8)$ and has its centre on the line $2x = y - 5$.

The centre of the circle lies on the perpendicular bisector of AB where A and B are the points $(4, -10)$ and $(-8, 8)$.

The middle point of AB is $(-2, -1)$

The slope of $AB = -\frac{1}{\frac{3}{2}} = -\frac{2}{3}$

$\therefore$ the slope of the $\perp$ bisector of $AB = \frac{3}{2}$

$\therefore$ the equation of the $\perp$ bisector of AB is

$$y + 1 = \frac{3}{2}(x + 2)$$

$$\text{i.e., } 3y + 3 = 2x + 4$$

$$\text{i.e., } 2x - 3y + 1 = 0$$

The centre lies on $2x = y + 5$ also.

$\therefore$ the centre is obtained by solving the two equations

$$2x - 3y + 1 = 0 \text{ and}$$

$$2x = y + 5.$$

We get $y = 3$ and $x = 4$

$\therefore$ the centre is $C(4, 3)$

The circle passes through $A(4, -10)$

$$\begin{aligned} \therefore \text{the radius of the circle} &= CA \\ &= \sqrt{13^2} \\ &= 13 \end{aligned}$$

$\therefore$ the equation of the circle is

$$(x - 4)^2 + (y - 3)^2 = 13^2$$

$$\text{i.e., } x^2 + y^2 - 8x - 6y - 144 = 0.$$

5 Find the equation of the tangent at the point $(4, 1)$ on the circle $x^2 + y^2 - 2x + 6y - 15 = 0$.

The centre of the circle is $C(1, -3)$.

If P is $(4, 1)$, the tangent at P to the circle is $\perp$ to CP .

The slope of $CP = \frac{4}{4}$

$\therefore$ the slope of the tangent $= -\frac{3}{4}$

$\therefore$ the equation of the tangent is $y - 1 = -\frac{3}{4}(x - 4)$

$$\text{i.e., } 4y - 4 = -3x + 12$$

$$\text{i.e., } 3x + 4y - 16 = 0.$$

6 Find the centre and radius of the circle $x^2 + y^2 - 14x + 6y + 9 = 0$. Also find the equation to that diameter of the circle which passes through $(2, 1)$ and the equation of the chord through $(2, 1)$ which is bisected by the diameter.

The equation of the circle is $x^2 + y^2 - 14x + 6y + 9 = 0$.

$\therefore$ its centre is $(7, -3)$ and radius $= \sqrt{7^2 + 3^2 - 9} = 7$.

The slope of the line through $(2, 1)$ and the centre $(7, -3)$ is $-\frac{4}{5}$

$\therefore$ its equation is $y - 1 = -\frac{4}{5}(x - 2)$

$$\text{i.e. } 5y - 5 = -4x + 8$$

$$\text{i.e. } 4x + 5y - 13 = 0.$$

The chord which is bisected at $(2, 1)$ is $\perp$ to this diameter.

$\therefore$ its equation is of the form $5x - 4y = a$

Since it passes through $(2, 1)$ we have, $10 - 4 = a$

i.e. $a = 6$

$\therefore$ the equation required is $5x - 4y = 6$.

Exercises

1 Find the equations of the circles whose centres and radii are given below:-

	Centre	Radius	Ans.
(a)	$(-2, -3)$	2	$x^2 + y^2 + 4x + 6y + 9 = 0$
(b)	$(3, -5)$	4	$x^2 + y^2 - 6x + 10y + 18 = 0$
(c)	$(6, -3)$	5	$x^2 + y^2 - 12x + 6y + 20 = 0$
(d)	$(0, -5)$	6	$x^2 + y^2 + 10y - 11 = 0$

2 Find the centres and radii of the following circles:-

(a)	$x^2 + y^2 - 6x + 8y - 24 = 0$	Ans. $(3, -4); 7$
(b)	$x^2 + y^2 - 8x + 7 = 0$	Ans. $(4, 0); 3$
(c)	$2(x^2 + y^2) + 6x - 10y + 9 = 0$	Ans. $(-\frac{3}{2}, \frac{5}{2}); 2$
(d)	$x^2 + y^2 - 8x - 6y - 24 = 0$	Ans. $(4, 3); 7$
(e)	$x^2 + y^2 + 6x - 4y - 12 = 0$	Ans. $(-3, 2); 5$

3 Show that the line $4x - y = 17$ passes through the centre of the circle $x^2 + y^2 - 8x + 2y = 0$.

4 Obtain the equation of the circle whose centre is at $(4, 5)$ and whose circumference passes through the centre of the circle $x^2 + y^2 + 4x + 6y = 12$.

$$\text{Ans. } x^2 + y^2 - 8x - 10y - 59 = 0$$

3 Find the equation of the circle two of whose diameters are $x + y = 6$ and $x + 2y = 4$ and whose radius is 10 units. Ans. $x^2 + y^2 - 16x + 4y - 32 = 0$

6 If one end of a diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$ be $(4, 1)$, find the coordinates of the other end. Ans. $(-2, -7)$

7 Show that the circles $x^2 + y^2 - 2x + 6y + 6 = 0$ and $x^2 + y^2 - 5x + 6y + 15 = 0$ touch each other internally.

[Hint:—Show that the distance between the centres is the difference of the radii].

8 Show that the circles $x^2 + y^2 + 2x - 4y - 4 = 0$ and $x^2 + y^2 - 4x + 4y + 4 = 0$ touch each other externally.

9 Find the equation to the circle passing through the points $(3, -4)$, $(-2, 1)$ and $(-3, -1)$.

Ans. $x^2 + y^2 - x + 3y - 10 = 0$

10 Find the equation to the circle passing through the points $(17, 3)$, $(-7, -7)$ and $(10, -14)$.

Ans. $x^2 + y^2 - 10x + 4y - 140 = 0$

11 Find the equation of the circle passing through the points $(0, 4)$ and $(4, 2)$ and having its centre on the line $3x + 4y = 7$. Ans. $x^2 + y^2 - 2x - 2y - 8 = 0$

12 Obtain the equation of the chord of the circle $x^2 + y^2 - 4x - 2y = 20$ which is bisected at the point $(2, 3)$.

Ans. $y = 3$

13 Find the equation of the circle drawn on the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter.

[Let $P(x, y)$ be any point on the circle. Then

$$PA \text{ is } \perp \text{ to } PB. \quad \therefore \frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1$$

i.e., $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, which is the equation of the required circle].

14 Find the equation of the circle described on the line joining the points $(-2, -1)$ and $(3, 4)$ as diameter.

Ans. $x^2 + y^2 - x - 3y - 10 = 0$

15 Show that the points $(1, 1)$, $(-2, 2)$, $(-2, -8)$ and $(-6, 0)$ are concyclic. Ans. $x^2 + y^2 + 4x + 6y - 12 = 0$

16 A circle has its centre on the x axis and passes thro' $(5, 1)$ and $(3, 4)$. Find its equation. $2x^2 + 2y^2 - x - 47 = 0$.

17 Show that the circle $x^2 + y^2 - 8x + 4y + 4 = 0$ touches the Y -axis.

18 Find the points of intersection of the circle $x^2 + y^2 - x + 3y - 10 = 0$ with the line $x + y + 1 = 0$.

Ans. $(3, -4)$, $(-2, 1)$

19 Find the points of intersection of the circle $x^2 + y^2 - 2x - 4y - 5 = 0$ with $3x + y = 15$. What is your inference? Ans. $(4, 3)$, $(4, 3)$

13 The Equation of the tangent to a circle at a given point

Let the equation of a circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Let P be the point (x_1, y_1) on it and let PT be the tangent to the circle at the point P .

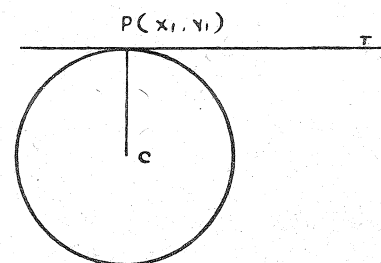


Fig. 8.

The centre is $C(-g, -f)$.

Slope of CP is $\frac{y_1 + f}{x_1 + g}$.

$\therefore$ the slope of PT is $-\frac{x_1 + g}{y_1 + f}$.

$\therefore$ the equation of PT is $y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$

i.e. $(x - x_1)(x_1 + g) = -(y - y_1)(y_1 + f)$
 i.e. $xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$
 Adding $gx_1 + fy_1 + c$ to both sides,
 we get $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
 $= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$, since (x_1, y_1) lies
 on the circle.

Hence the required equation is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Worked Examples

- Find the equation of the tangent to the circle $x^2 + y^2 = 85$ at the point $(7, 6)$.
 The required tangent is given by the equation
 $x(7) + y(6) = 85$, i.e. $7x + 6y = 85$.
- Find the equation of the tangent at the point $(-1, 6)$
 on the circle $x^2 + y^2 + 8x + 10y - 89 = 0$.
 The tangent is $x(-1) + y(6) + 4(x - 1) + 5(y + 6)$
 $- 89 = 0$,
 i.e. $3x + 11y - 63 = 0$.

Exercises

- 1 Find the equation of the circle which passes through the origin and the points $(6, 0)$ and $(0, 8)$. Also find the equation of the tangent to the circle at the origin.

Ans. $x^2 + y^2 - 6x - 8y = 0$; $3x + 4y = 0$

- 2 One vertex of a square is the origin and two others are $(4, 0)$ and $(0, 4)$. Find the equation to the circle circumscribing the square. Also find the equation to the tangent to this circle at the origin.

Ans. $x^2 + y^2 - 4x - 4y = 0$; $x + y = 0$

- 3 Find the equation of the tangent to the circle $x^2 + y^2 - 10x + 4y - 140 = 0$ at the point $(-7, -7)$ on it.

Ans. $12x + 5y + 119 = 0$

- 4 Find the equation of the tangent to the circle $x^2 + y^2 - x + 3y - 10 = 0$ at the point $(-2, 1)$. Ans. $x - y + 3 = 0$

- 5 Find the equation of the circle described on the line joining the points $A(-3, 7)$ and $B(2, -5)$ as diameter.

Also obtain the equations of the tangents of the above circle at A and B .

Ans. $x^2 + y^2 + x - 2y - 4 = 0$; $5x - 12y + 99 = 0$;
 $5x - 12y - 70 = 0$

- 6 Find the equation of the circle described on the line joining the points $(3, 5)$ and $(4, -1)$ as diameter. Find the equations to the tangents at these points to the circle.

Ans. $x^2 + y^2 - 7x - 4y + 7 = 0$. $x - 6y + 27 = 0$
 $x - 6y - 10 = 0$

- 7 Find the condition that the straight line $y = mx + c$ may be a tangent to the circle $x^2 + y^2 = a^2$.

Ans. $c^2 = a^2(1 + m^2)$

- 8 Find the condition that $ax + by + c = 0$ may be a tangent to the circle $(x - h)^2 + (y - k)^2 = r^2$.

Ans. $(ah + bk + c)^2 = r^2(a^2 + b^2)$

- 9 Find the equations to the tangents to the circle $x^2 + y^2 = 25$ which are i) parallel ii) perpendicular to, $4x - 3y = 0$.

Ans. $4x - 3y = \pm 25$; $3x + 4y = \pm 25$.

- 10 Find the equations to the tangents to the circle $x^2 + y^2 - 8x + 12 = 0$ which are parallel to the line $3x - 4y = 0$.

Ans. $3x - 4y - 22 = 0$, $3x - 4y - 2 = 0$.

- 14 The length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from an external point (x_1, y_1)

Let A be the point (x_1, y_1)

Let C be the centre of the given circle.

Let AT be the tangent to the circle touching the circle at T .

Join CT .

We know that C is $(-g, -f)$ and that the radius of the circle is $\sqrt{g^2 + f^2 - c}$.

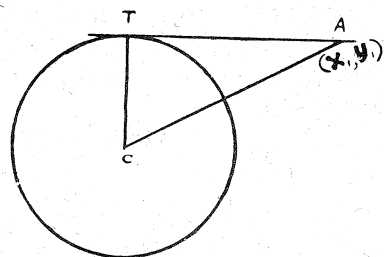


Fig. 9

Since $\angle ATC = 90^\circ$, we have

$$\begin{aligned} AT^2 &= AC^2 - CT^2 \\ &= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \end{aligned}$$

$\therefore$ the length of the tangent to the circle from the point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

Worked Examples

1 Find the length of the tangent to the circle $x^2 + y^2 + 8x - 6y + 24 = 0$, drawn from the point $(1, 2)$.

If t is the length of the tangent, then

$$\begin{aligned} t^2 &= (1)^2 + (2)^2 + 8(1) - 6(2) + 24 \\ &= 25 \end{aligned}$$

$$\therefore t = 5.$$

2 Find the length of the tangent to the circle $2(x^2 + y^2) - 3x + 4y + 6 = 0$, from the point $(4, 1)$.

The equation in the standard form is

$$x^2 + y^2 - \frac{3}{2}x + 2y + 3 = 0.$$

If t is the length of the tangent, then

$$\begin{aligned} t^2 &= (4)^2 + (1)^2 - \frac{3}{2}(4) + 2(1) + 3 \\ &= 16 \end{aligned}$$

$$\therefore t = 4.$$

Exercises

1 Find the length of the tangent from $(5, 7)$ to the circle $x^2 + y^2 - 4x + 4y - 8 = 0$.
Ans. $\sqrt{74}$

2 Find the length of the tangent from $(-1, 1)$ to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$.
Ans. $\sqrt{13}$

3 Find the locus of the point which moves so that the lengths of the tangents drawn from it to the circles $x^2 + y^2 - y - 1 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ are in the ratio of 1:2.
Ans. $3(x^2 + y^2) + 6x - 4y - 9 = 0$

4 Find the locus of a point which moves so that the length of the tangent from it to the circle $x^2 + y^2 = 25$ is equal to its distance from the line $3x - 4y = 0$.
Ans. $16x^2 + 9y^2 + 24xy - 625 = 0$

5 Show that the lengths of the tangents from the point $(1, 5)$ to the circles $x^2 + y^2 + 4x + 2y - 3 = 0$ and $x^2 + y^2 - 6x + 4y - 3 = 0$ are equal.

6 Find the equation of the circle such that the lengths of the tangents from the points $(-1, 0)$, $(0, 2)$ and $(-2, 1)$ may be 3, $\sqrt{10}$ and $3\sqrt{3}$ respectively.
Ans. $x^2 + y^2 - 10x + 4y - 2 = 0$

7 Find the point on the line $x - y = 4$, from which the tangents to the circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 2x + 8y - 7 = 0$ are of equal lengths.
Ans. $(5, 1)$

15 Power of a point with respect to a circle

The Power of a point with respect to a circle, centre C and radius r , is defined to be the value of $CP^2 - r^2$. This will be positive, negative or zero according as the point P is outside, inside or on the circle.

If the point P is (x_1, y_1) and the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, then the power of P with respect to the circle $= CP^2 - r^2$

$$\begin{aligned} &= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \end{aligned}$$

Note: If P is outside the circle then its power is equal to the square of the tangent from it to the circle.

16 Radical Axis of two circles

The locus of a point, powers of which with respect to two circles are equal is called the **Radical Axis** of the two circles.

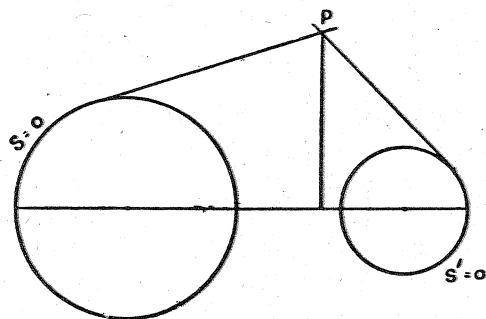


Fig. 10

Let the circles be $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Let P be the point (x_1, y_1) .

The powers of P with respect to the two circles are equal.

$$\therefore x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2.$$

$$\text{i.e. } 2x_1(g_1 - g_2) + 2y_1(f_1 - f_2) + c_1 - c_2 = 0$$

$$\therefore \text{the locus is, } 2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0.$$

Note: 1. If the equations of the two circles in the standard form are $S = 0$ and $S' = 0$, then the Radical Axis is $S - S' = 0$.

2. The locus is of the first degree in x and y and is therefore a straight line.

3. The Radical Axis is perpendicular to the line of centres.

The centres of the circles are $(-g_1, -f_1)$ and $(-g_2, -f_2)$

$$\text{The slope of the line of centres} = \frac{f_1 - f_2}{g_1 - g_2}$$

$$\text{The slope of the Radical Axis} = -\frac{g_1 - g_2}{f_1 - f_2}$$

$\therefore$ the product of the slopes $= -1$. Hence the two are perpendicular to each other.

4. If the two circles cut at P and Q , then the tangents from P and Q to the circles are of zero lengths and are therefore equal.

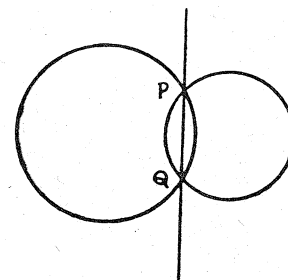


Fig. 11

Hence P and Q must lie on the Radical Axis. Hence the Radical Axis is the line PQ , i.e. the common chord of the two circles.

17 The equations of two circles in their simplest form

Choose the line of centres as the X -axis and the Radical Axis as the Y -axis. Then the centres of the two circles are on the X -axis and therefore the y -coordinates of the centres are zeroes.

$\therefore$ the equations of the circles are of the form $x^2 + y^2 + 2g_1x + c_1 = 0$ and $x^2 + y^2 + 2g_2x + c_2 = 0$.

Also the powers of the origin with respect to the two circles are c_1 and c_2 . These must be equal because the origin is a point on the Radical Axis. Hence $c_1 = c_2$.

Therefore the equations can be taken as $x^2 + y^2 + 2g_1x + c = 0$ and $x^2 + y^2 + 2g_2x + c = 0$ where the constant c is the same in both cases.

- 18 The Radical Axes of three circles taken in pairs are concurrent

Let the equations of the three circles be $S=0$, $T=0$, and $U=0$ in their standard forms. S , T and U are expressions of the form $x^2 + y^2 + 2g_1x + 2f_1y + c_1$, $x^2 + y^2 + 2g_2x + 2f_2y + c_2$ and $x^2 + y^2 + 2g_3x + 2f_3y + c_3$ respectively.

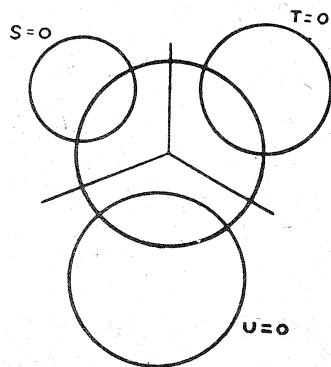


Fig. 12

Then the Radical axes of the three circles taken in pairs are given by $S-T=0$, $T-U=0$, and $U-S=0$.

It is clear that $(S-T) + (T-U) + (U-S) = 0$ identically.

Hence by Art 9, the three lines are concurrent.

Note: The point of concurrence of the three Radical Axes is called the **Radical Centre** of the three circles.

The powers of the Radical Centre with respect to the three circles are equal.

If the centre lies outside the circles, the lengths of the tangents to the three circles from the Radical Centre are equal.

Exercises

- 1 Find the Radical Axis of the circles,
 $x^2 + y^2 + 2x + 2y + 1 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$.
 Ans. $12x + 8y - 13 = 0$

- 2 Find the Radical Axis of the circles
 $x^2 + y^2 + 4x - 6y - 2 = 0$ and $x^2 + y^2 - 2x + 8y + 1 = 0$
 and find the point of intersection with the line of centres.

Ans. $6x - 14y - 3 = 0$; $\left(-\frac{61}{116}, \frac{51}{116}\right)$

- 3 Find the Radical Centre of the three circles
 $x^2 + y^2 - 6x - 6y + 4 = 0$, $x^2 + y^2 - 2x - 4y + 3 = 0$, and
 $x^2 + y^2 + 2x + 2y + 1 = 0$. Show that it lies outside the
 circles.

Ans. $\left(\frac{1}{3}, \frac{1}{3}\right)$; power + ve

- 4 Find the Radical Centre of the three circles
 $3(x^2 + y^2) = 1$, $2(x^2 + y^2) + x = 2$, $x^2 + y^2 + x + y = 3$. Find
 the length of the tangent from it to the circles.

Ans. $\left(\frac{4}{3}, \frac{4}{3}\right)$; $\frac{\sqrt{29}}{3}$

- 5 Find the Radical Axis of the circles
 $x^2 + y^2 - 4x + 6y - 10 = 0$ and $x^2 + y^2 + 2x - 6y + 2 = 0$
 and show that the circles intersect in real points.

Ans. $x - 2y + 2 = 0$

19 Orthogonal Circles

Definition If the tangents at a common point of two intersecting circles are at right angles, the circles are said to cut orthogonally.

Let circles whose centres are A and B cut orthogonally at P .

The tangents at P to the two circles are at right angles.

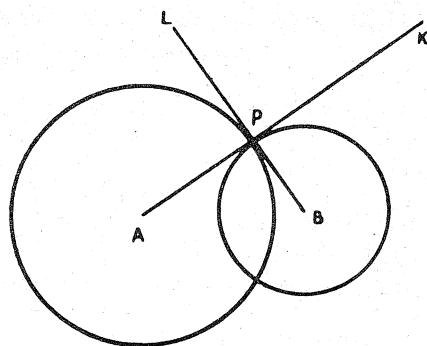


Fig. 13

Let the tangents be PK and PL .

Then angle $KPL = 90^\circ$.

PL is a tangent to circle centre A , and PK is perpendicular to PL .

$\therefore PK$ must be a radius of circle, centre A i.e. PK must pass through A .

Similarly PK is a tangent to the circle centre B , and PL is perpendicular to PK .

$\therefore PL$ must pass through B .

In other words the radii PA and PB must be at right angles.

20 The condition that two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ may cut orthogonally

If the two circles cut orthogonally, then $\angle APB = 90^\circ$ where A and B are the centres of the circles and P is a point of intersection.

$$\therefore AB^2 = AP^2 + PB^2.$$

A is $(-g_1, -f_1)$ and B is $(-g_2, -f_2)$

$$AB^2 = (g_1 - g_2)^2 + (f_1 - f_2)^2$$

$$AP^2 = g_1^2 + f_1^2 - c_1, \quad PB^2 = g_2^2 + f_2^2 - c_2$$

$$\therefore (g_1 - g_2)^2 + (f_1 - f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$

i.e. $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

21 The circle with any point as centre and the length of the tangent from it to a given circle as radius cuts the given circle orthogonally.

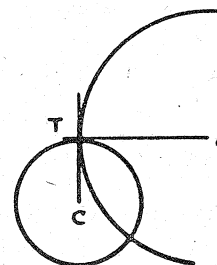


Fig. 14

Let C be the centre of the given circle and P the given point.

Let PT be the tangent to the given circle. Draw the circle with P as centre and radius PT .

Join CT .

CT is a radius of circle, centre C and TP is the tangent at T . Hence $\angle CTP = 90^\circ$.

PT is the radius of the other circle.

Hence the radii of the two circles through T are at right angles.

Hence the two circles cut orthogonally.

Note: 1. The circle with any point on the Radical Axis of two circles as centre and the length of the tangent from the point to either of the two circles as radius cuts both the circles orthogonally.

2. The circle with the radical centre of three circles as centre and the length of the tangent from it to any of the three circles as radius cuts the three circles orthogonally. This is called the **Orthotomic** circle of the three circles, (Radical circle or common orthogonal circle).

Worked Examples

1 Find the circle whose centre is (1, 2) and which cuts the circle $x^2 + y^2 - 4x + 6y - 2 = 0$ orthogonally.

The radius of the circle = the length of the tangent from the point (1, 2) to the circle

$$= \sqrt{1^2 + 2^2 - 4 + 12 - 2} \\ = \sqrt{11}$$

$$\therefore \text{ the circle is } (x-1)^2 + (y-2)^2 = 11$$

$$\text{i.e. } x^2 + y^2 - 2x - 4y - 6 = 0.$$

2 Find the equation of the circle cutting the two circles $x^2 + y^2 + 6x + 2y + 1 = 0$ and $x^2 + y^2 + 2x + 4y - 5 = 0$ orthogonally and having its centre on the line $3x + 2y - 6 = 0$.

The Radical Axis of the two circles is $4x - 2y + 6 = 0$,
i.e. $2x - y + 3 = 0$

The centre of the required circle lies on this R.A. and also on $3x + 2y = 6$

Solving, we get, $x = 0, y = 3$

$\therefore$ its centre is (0, 3)

The radius of the required circle is the length of the tangent from (0, 3) to either of the circles.

$$\text{i.e. the radius} = \sqrt{9 + 6 + 1} \\ = 4$$

$$\therefore \text{ the equation required is } x^2 + (y-3)^2 = 16$$

$$\text{i.e. } x^2 + y^2 - 6y - 7 = 0.$$

3 Find the circle which cuts the three circles $x^2 + y^2 + 5x - 2y + 4 = 0$, $x^2 + y^2 + 2x + 4y + 1 = 0$ and $x^2 + y^2 + 4x + 4y - 1 = 0$ orthogonally.

The R. A. of circles (2) and (3) is $-2x + 2 = 0$

$$\text{i.e. } x = 1.$$

The R.A. of circles (1) and (2) is $3x - 6y + 3 = 0$.

The centre of the required circle lies on both these Radical Axes and is got by solving between these two equations.

Solving we get, $x = 1, y = 1$.

$\therefore$ the centre is (1, 1)

The radius of the required circle is the length of the tangent from (1, 1) to any one of the three circles,

$$\text{i.e. the radius} = \sqrt{1 + 1 + 5 - 2 + 4} = 3$$

$$\therefore \text{ the equation required is } (x-1)^2 + (y-1)^2 = 9$$

$$\text{i.e. } x^2 + y^2 - 2x - 2y - 7 = 0.$$

4. Find the equation of the circle passing through (1, 2), having its centre on $2x + y = 1$ and cutting orthogonally the circle $x^2 + y^2 - 2x + 1 = 0$.

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

It passes through (1, 2), $\therefore 1 + 4 + 2g + 4f + c = 0$

$$\text{i.e. } 2g + 4f + c = -5 \dots (1)$$

the centre $(-g, -f)$ lies on $2x + y = 1$

$$-2g - f = 1 \dots (2)$$

It cuts the given circle orthogonally

$$-2g = c + 1 \dots (3)$$

Solving between these three equations we get, $g = 0$
 $f = -1, c = -1$

$\therefore$ the equation of the required circle is
 $x^2 + y^2 - 2y - 1 = 0.$

Exercises

① Show that the circles $x^2 + y^2 + 2x - 4y + 5 = 0$ and $x^2 + y^2 - 6x + 4y - 19 = 0$ cut orthogonally.

② Find the value of c for which the circles $x^2 + y^2 - x + 22y + 3 = 0$ and $x^2 + y^2 + 7x + 6y + c = 0$ are orthogonal.

$$\text{Ans. } c = \frac{119}{2}$$

③ Find the equation of the circle which passes through the origin and cuts orthogonally the circles $x^2 + y^2 + 8x + 4y - 6 = 0$ and $x^2 + y^2 + 2x + 6y + 1 = 0$.

$$\text{Ans. } x^2 + y^2 - 2x + y = 0$$

④ Find the equation of the circle that passes through the origin, has its centre on $x + y = 4$ and cuts orthogonally the circle $x^2 + y^2 - 4x + 2y + 4 = 0$.
Ans. $x^2 + y^2 - 4x - 4y = 0$

5 Find the equation of the circle cutting the circles $x^2 + y^2 + 2x + 17y + 5 = 0$ and $x^2 + y^2 + 7x + 6y + 11 = 0$ orthogonally and having its centre on $2x + 3y = 5$.

Ans. $x^2 + y^2 - 2x - 2y - 24 = 0$

6 Find the equation of the circle cutting orthogonally the circles $3(x^2 + y^2) = 1$, $2(x^2 + y^2) + x = 2$, and $x^2 + y^2 + x + y = 3$.

Ans. $3(x^2 + y^2) - 8x - 8y + 1 = 0$

7 Find the locus of the centre of a circle which cuts the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ orthogonally

Ans. $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

8 Find the equation of the circle which cuts orthogonally the circles $x^2 + y^2 + 2x - 2y - 2 = 0$ and $x^2 + y^2 + 6x - 4y + 4 = 0$ and has its centre on the line $4x - 3y + 3 = 0$.

Ans. $x^2 + y^2 + 6x + 6y + 2 = 0$

9 Show that the equation of a circle cutting orthogonally the circles $x^2 + y^2 + 2g_1x + c = 0$ and $x^2 + y^2 + 2g_2x + c = 0$ is of the form $x^2 + y^2 + 2fy - c = 0$.

10 Find the equations of the two circles which cut the circles $x^2 + y^2 - 6y + 1 = 0$, $x^2 + y^2 - 4y + 1 = 0$ orthogonally and touch the line $3x + 4y + 5 = 0$.

Ans. $x^2 + y^2 = 1$; $4(x^2 + y^2) - 15x - 4 = 0$

CALCULUS

CHAPTER I

FUNCTIONS

1 Functions Given two sets A and B , if every element of A is associated with some unique element of B we say that A is related to B by a function. We write this in the form $f: A \rightarrow B$ and read it as ' f is a function of A into B '. The set A is called the Domain of the function and the set B is called the Range of the function f . If $a \in A$ and the corresponding element in B is b , then we call b the image of a . These ideas have been already dealt with in an earlier chapter (Vol. I, Chapter III).

2 One-one function Let f map A into B , i.e. every element of A is assigned to some unique element of B . If in addition to this, different elements in B are assigned to different elements in A , then the function is said to be a

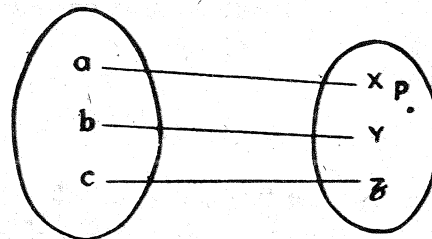


Fig. 1

one-one function. It means that no two different elements in A are assigned to the same element in B . An element in the Range is matched with only one element in the Domain or no element in the Domain.

Examples

1) Let the function $f: A \rightarrow B$ be defined by the formula $f(x) = x^2$, where A and B are sets of real numbers. For every number in A there is a unique number in B . Hence it is a function. But for every number in B the corresponding number in A is not unique, e.g., $f(2) = 4$, $f(-2) = 4$ i.e. the element in A which corresponds to the element 4 in B is not unique. Hence it is not an one-one function.

2) Consider the function $f: A \rightarrow B$, where the function is defined by the formula $f(x) = x^3$, A and B being sets of real numbers. Here every element of A has got a unique image in B and every number in B has got a unique element in A . Hence it is an one-one function.

3 Onto functions Let f be a function of A into B i.e. every element of A is assigned to some unique element of B . Further, if every element of B is the image of at least one member of A , we call this an onto function. This means that if corresponding elements of A and B are written down, every element of B must be represented and must be associated with at least one element of A . We also write this as ' f is a function of A onto B ' or ' f maps A onto B '.

Examples

1- Let us take the function $f: A \rightarrow B$, defined by the formula $f(x) = x^2$, A and B being real. The negative numbers in B are not the squares of any number in A , they do not appear in the Range of f . Hence this is not an onto function.

2 Let us consider the sets $A = \{a, b, c, d\}$ and $B = \{p, q, r\}$. Let the function f of A into B be defined by the

following: $f(a) = p$, $f(b) = p$, $f(c) = q$, and $f(d) = q$. This is not an onto function as the element r of B is not associated with any element of A .

3) Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$. Let $f: A \rightarrow B$ be defined by $f(d) = 1$, $f(a) = f(b) = 2$, $f(c) = 3$. This is an onto function because every element of B is the image of at least one element of A .

Exercises

① Let $A = \{a, b, c, d\}$ and let $B = \{p, q, r, s, t, u, v\}$. If $f(a) = p$, $f(b) = q$, $f(c) = r$, $f(d) = s$, is the function one-one? Ans. yes

② If in the above we have, i) $f(a) = p$, $f(b) = q$, $f(c) = p$, $f(d) = r$, is the function one-one? Ans. no

③ Let $A = \{x \mid -2 \leq x \leq 2\}$ and let $f: A \rightarrow B$ where A and B are real, defined by the formula $f(x) = x^2$. Is the function one-one? Ans. no

④ Let $A = \{x \mid -4 \leq x \leq -1\}$ and let $f: A \rightarrow B$ where A and B are real, defined by the formula $f(x) = x^2$. Is the function one-one? Ans. yes

⑤ Let $A = \{x \mid 1 \leq x \leq 4\}$ and let $f: A \rightarrow B$, A and B being real, defined by the formula $f(x) = x^2$. Is the function one-one? Ans. yes

⑥ Let $A = \{1, 2, 3\}$ and let $B = \{p, q\}$. In how many ways can functions be formed from A into B ? How many of these are one-one? Ans. 8; none

⑦ Let $A = \{1, 2, 3, 4\}$ and let $B = \{1, 2, 3, 5\}$. A function is defined by $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $f(4) = 1$. Is the function an onto function? Ans. no

⑧ Let $A = \{a, b, c\}$ and let $B = \{1, 4\}$. In how many ways can onto functions be formed from A into B ? Ans. 6

⑨ If $A = \{x \mid x \in R \text{ and } \neq 0\}$ prove that $f: A \rightarrow A$ defined by $f(x) = \frac{1}{x}$ is a one-one and onto function.

4 Inverse functions Let $f: A \rightarrow B$. We define the inverse of any element b of B as the set of those elements of A which have b as their image, e.g. if $A = \{a, b, c\}$ and $B = \{p, q, r\}$ and if a function is defined as $f(a) = p$, $f(b) = q$, $f(c) = r$, then the inverses of p, q, r are respectively $\{a\}, \{b\}, \{c\}$.

Again let $A = \{p, q, r\}$ and let $B = \{x, y, z\}$. If $f: A \rightarrow B$ is defined by the assignments $f(p) = x$, $f(q) = x$, $f(r) = z$, then the inverse of x is $\{p, q\}$, and the inverse of z is $\{r\}$. The inverse of y is the null set since there is no element of A corresponding to y .

Inverses are generally represented by the symbol f^{-1} . Thus in the above example we can write $f^{-1}(x) = \{p, q\}$, $f^{-1}(y) = \phi$. We can therefore say that if $f: A \rightarrow B$ ($a \in A$, $b \in B$) then $f^{-1}(b) = \{x \mid x \in A, f(x) = b\}$.

Let us consider the function $f: A \rightarrow B$ where $f(x) = x^2$, and $A = \{x \mid x \text{ real}\}$ and $B = \{y \mid y \text{ real}\}$. Then $f^{-1}(9) = \{3, -3\}$, $f^{-1}(16) = \{4, -4\}$ etc. Note that $f^{-1}(-5) = \phi$ since there is no element in A corresponding to -5 in B . In general the inverses of the elements of B can be more than one element or even the empty set.

But if $f: A \rightarrow B$ is an one-one and onto function then the inverse of every element of B will be a single element of A , i.e. f^{-1} is a function of B into A . This function is called the **inverse function** of f . The inverse function exists only if $f: A \rightarrow B$ is one-one and also onto.

Note: The student must note the difference between the terms 'Inverse of a function' and 'Inverse function'.

Worked Examples

1) If $f: A \rightarrow B$ is given by $y = f(x) = 2x + 1$, find $f^{-1}(x)$.

f is one-one and onto. Hence f has an inverse function f^{-1} .

$$y = f(x) = 2x + 1$$

$$\therefore x = \frac{y-1}{2}$$

i.e. $f^{-1}(y) = \frac{y-1}{2}$ defines the inverse function.

It is customary to use the variable x also for the inverse function.

Hence the inverse function $= \frac{x-1}{2}$

2) If $f: R \rightarrow R$ is given by $f(x) = x^2 + 4$, find $f^{-1}(8)$.

$$\text{Let } f^{-1}(8) = y.$$

$$\text{Then } f(y) = 8,$$

$$\text{i.e. } y^2 + 4 = 8$$

$$\text{i.e. } y^2 = 4$$

$$\text{i.e. } y = -2, 2.$$

$$\therefore f^{-1}(8) = \{-2, 2\}.$$

3) If $f: R \rightarrow R$ is given by $f(x) = x^2 - 3x + 4$, find $f^{-1}(2)$.

$$\text{Let } f^{-1}(2) = y$$

$$\text{Then } f(y) = 2$$

$$\text{i.e. } y^2 - 3y + 4 = 2$$

$$\text{i.e. } y^2 - 3y + 2 = 0$$

$$\text{i.e. } (y-1)(y-2) = 0$$

$$\text{i.e. } y = 1, 2, \therefore f^{-1}(2) = \{1, 2\}.$$

Exercises

① Let $f: A \rightarrow B$ where $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$. If $f(1) = 1$, $f(2) = 5$, $f(3) = 3$, $f(4) = 3$, find $f^{-1}(5)$, $f^{-1}(3)$, $f^{-1}(1)$.
Ans. $\{2\}, \{3, 4\}, \{1\}$

② Let $f: A \rightarrow B$. Let $A = \{2, 4, 6, 9\}$ and $B = \{p, q, r, s\}$ find $f^{-1}(p)$, $f^{-1}(q)$, $f^{-1}(r)$, $f^{-1}(s)$, given that $f(2) = p$, $f(4) = p$, $f(6) = p$, $f(9) = q$.
Ans. $\{2, 4, 6\}, \{9\}, \phi, \phi$

③ If $f: R \rightarrow R$ where R represents real numbers and is defined by the formula $f(x) = x^2$, find $f^{-1}(49)$, $f^{-1}(-16)$.
Ans. $\{7, -7\}, \phi$

④ If elements in A are assigned to corresponding elements in B as in the following figures, state the nature of the relationship between them:

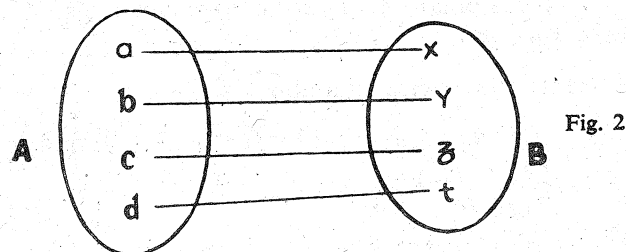


Fig. 2

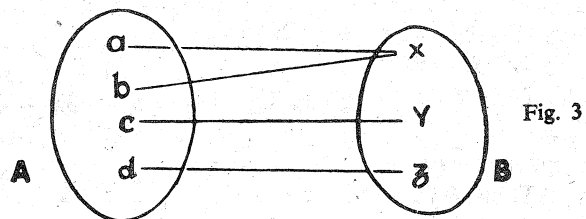


Fig. 3

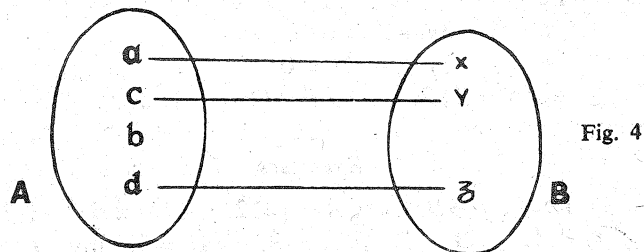


Fig. 4

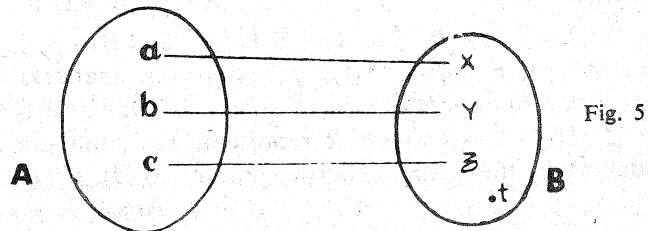


Fig. 5

Ans. i) one one and onto, ii) not one-one but onto iii) not a function iv) not onto but one-one; only i) has an inverse

⑤ Let $A = \{x \mid -1 \leq x \leq 1\}$. Let $f(x)$ be the function of A into A defined by i) $f(x) = x^2$, ii) $f(x) = x^3$. State which of these has an inverse.

Ans. i) neither one-one nor onto

ii) both one-one and onto; ii) has an inverse.

⑥ Let $f: A \rightarrow B$ be defined by the formula $f(x) = 3x - 4$, A and B being real numbers. Has f an inverse? What is the formula for the inverse?

Ans. yes; $\frac{x+4}{3}$.

⑦ Let $f: A \rightarrow B$ be defined by $f(x) = 3x^3 - 7$. Is the function one-one and onto? Has it an inverse? If so, find the formula.

Ans. one-one and onto; $\left(\frac{x+7}{3}\right)^{\frac{1}{3}}$

⑧ Let $A = \{x, y, z\}$ and $B = \{1, 2, 3\}$. If $f: A \rightarrow B$, is defined by the following functions, state whether they are one-one and onto:

1) $f(x) = 1, f(y) = 2, f(z) = 3$.

Ans. one-one and onto

2) $f(x) = 1 = f(y), f(z) = 2$. not one-one, not onto

⑨ If $A = \left\{x \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\}$ and $B = \{y \mid -1 \leq y \leq 1\}$, and if $f: A \rightarrow B$ is defined by $y = \sin x$, state whether the function is one-one and onto.

Ans. yes

⑩ If $A = \{x \mid x \in \mathbb{R}, x \neq 2\}$, prove that $f: A \rightarrow B$ defined by $f(x) = \frac{1}{x-2}$ is one-one and onto.

⑪ If $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = 7x - 8$, find $f^{-1}(x)$ and $f^{-1}(6)$.

Ans. $\frac{x+8}{7}; \{2\}$

⑫ If $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = x^2 + 3$, find $f^{-1}(7)$ and $f^{-1}(12)$.

Ans. $\{-2, 2\}; \{-3, 3\}$

13 If $f: R \rightarrow R$ is defined by $f(x) = x^2 - 5$, find $f^{-1}(20)$ and $f^{-1}(-12)$ Ans. $\{-5, 5\}; \emptyset$

14 If $f: R \rightarrow R$ is defined by $f(x) = x^2 - 10x + 21$, find $f^{-1}(-3)$. Ans. $\{4, 6\}$

15 If $f: C \rightarrow C$ is defined by $f(x) = x^2$, find $f^{-1}(-9)$. Ans. $\{-3i, +3i\}$

[C is the set of complex numbers].

5 Composite functions

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be any two functions. Let $a \in A$. Then to this element in A there corresponds an element $b = f(a)$ in B . To this element in B there corresponds an element $c = g(b)$ in C . Therefore to the element a in A there corresponds an element c in C . That is $c = g(b) = g[f(a)]$. We have thus a function of A into C . This function is called the Composite function (or Product function) of f and g . This Composite function is denoted by $(g \circ f)$ and can be diagrammatically represented as in fig. below:

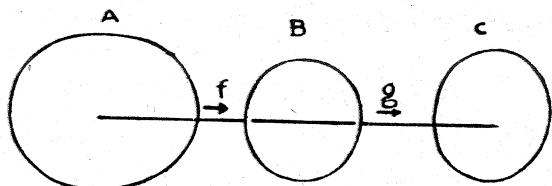


Fig. 6

Worked Examples

1) Let $f(x) = x^2$ and let $g(x) = x + 7$. Form the composite functions $(f \circ g)(3)$ and $(g \circ f)(3)$ and show that they are not the same.

$$g(3) = 10; \text{ hence } (f \circ g)(3) = f(10) = (10)^2 = 100.$$

$$f(3) = 9; \text{ hence } (g \circ f)(3) = g(9) = 9 + 7 = 16.$$

They are not equal.

2) Let $f: A \rightarrow B$ given by $f(x) = 2x + 3$ and let $g: B \rightarrow C$ given by $g(x) = x^2 - 3$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = f(x^2 - 3) = 2(x^2 - 3) + 3 = 2x^2 - 3.$$

$$(g \circ f)(x) = g(2x + 3) = (2x + 3)^2 - 3 = 4x^2 + 12x + 6.$$

Exercises

1 If f and g are two functions given by $f(x) = x^2 + x + 1$ and $g(x) = 4x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$\text{Ans. } 16x^2 - 4x + 1; 4x^2 + 4x + 3$$

2 If $f: R \rightarrow R$ be defined by $f(x) = x^2 + 2$ for all $x \in R$ and $g: R \rightarrow R$ be defined by $g(x) = 2x - 1$ for all $x \in R$ form the functions $(g \circ f)$ and $(f \circ g)$.

$$\text{Ans. } 2x^2 + 3; 4x^2 - 4x + 3$$

3 Let $f: A \rightarrow B$ where A and B are real numbers and let f be defined by $f(x) = x^2 + x + 1$. Find $f(x^2)$ and $(f \circ f)(x)$. Ans. $x^4 + x^2 + 1; x^4 + 2x^3 + 4x^2 + 3x + 3$

4 Let f and g be two functions defined by $f(x) = 2x + 1$ and $g(x) = 4x^2$. Find $(f \circ g)$, $(g \circ f)$, $(f \circ f)$, $(g \circ g)$.

$$\text{Ans. } 8x^2 + 1; 4(2x + 1)^2; 4x + 3; 64x^4$$

5 Let f and g be two functions defined by $f(x) = 3x + 5$ and $g(x) = 2x - 5$. Find $f(2)$, $g(3)$, $(f \circ g)(2)$, $(g \circ f)(3)$. Ans. 11; 1; 2; 23

6 Consider the expression $y = \frac{x^2-1}{x-1}$. Giving to x various values like 2, 3, 4, ... the corresponding values of y can be obtained easily. There is a difficulty however when we give the value $x=1$ and try to find the corresponding value of y ; y then becomes $\frac{0}{0}$, which is meaningless. In this case, $y=f(x)$ exists for all values of x except when $x=1$. So long as $x \neq 1$, $y = \frac{x^2-1}{x-1} \equiv x+1$, and the relation between x and y is quite clear.

Let us give values to x close to 1, say .5, .9, .99, .999 etc. The corresponding values of $\frac{x^2-1}{x-1} \equiv x+1$ are 1.5, 1.9, 1.99, 1.999 etc., for, now we can divide by $x-1$ as $x \neq 1$. Thus it is clear that as we go on giving values to x closer and closer to 1, the corresponding values of y get closer and closer to 2.

In fact, it is evident that $\frac{x^2-1}{x-1}$ can be made as close to 2 as necessary by choosing a suitable value of x close enough to 1. This is otherwise stated in this way: $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = 2$.

This is read, 'limit of $\frac{x^2-1}{x-1}$ as x tends to 1 is 2'.

If we draw the graph of $y = \frac{x^2-1}{x-1}$, we can plot as many points as we like, by giving values to x like 0, 2, 3, ... and joining these points. This will be identical with the line $y = x+1$, except for the point where $x=1$.

The graph of $y = \frac{x^2-1}{x-1}$ will not contain a point on it corresponding to $x=1$, and we say that it has a discontinuity at the point, when $x=1$.

7 To find $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}$.

Case I. Let n be a positive integer.

Dividing $x^n - a^n$ by $x - a$, we have,

$$\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-1}.$$

This division is possible since $x - a \neq 0$.

When $x \rightarrow a$, each term on the right tends to a^{n-1} and there are n such terms.

$$\therefore \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

Case II. Let n be a positive fraction $\frac{p}{q}$, where p and q are positive integers.

$$\text{To find } \lim_{x \rightarrow a} \frac{x^{\frac{p}{q}} - a^{\frac{p}{q}}}{x - a}.$$

$$\text{Put } x^{\frac{1}{q}} = y \text{ and } a^{\frac{1}{q}} = b$$

$$\text{Then } x = y^q \text{ and } a = b^q.$$

Also, if $x \rightarrow a$, evidently $y \rightarrow b$.

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^{\frac{p}{q}} - a^{\frac{p}{q}}}{x - a} &= \lim_{y \rightarrow b} \frac{y^p - b^p}{y^q - b^q} \\ &= \frac{\lim_{y \rightarrow b} \frac{y^p - b^p}{y - b}}{\lim_{y \rightarrow b} \frac{y^q - b^q}{y - b}} \\ &= \frac{pb^{p-1}}{qb^{q-1}} \quad \because p \text{ and } q \text{ are +ve integers.} \\ &= \frac{p}{q} b^{p-q} \quad (\text{Case I}) \\ &= \frac{p}{q} \left(\frac{1}{a^{\frac{1}{q}}} \right)^{p-q} \\ &= \frac{p}{q} a^{\frac{p}{q} - 1} = na^{n-1}. \end{aligned}$$

From cases I and II, we can say that $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$ when n is any positive quantity.

Case III. Let n be a negative quantity.

Let $n = -m$, where m is therefore a positive quantity.

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} \frac{x^{-m} - a^{-m}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{\frac{1}{x^m} - \frac{1}{a^m}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{\frac{a^m - x^m}{x^m a^m}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x^m - a^m)}{x^m a^m} \cdot \frac{1}{x - a} \\ &= -ma^{m-1} \frac{1}{a^{2m}} \quad \because m \text{ is } +ve, \\ &= -ma^{-m-1} = na^{n-1}. \end{aligned}$$

Thus, whatever rational value n may have,

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

8 To find $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$, θ being measured in radians

Let OA and OB be two radii of a circle, centre O . Let $\angle AOB$ be a small angle θ , expressed in radian measure.

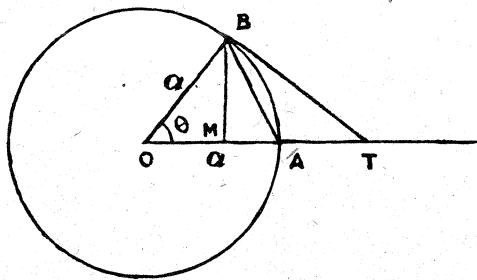


Fig. 9

Join AB . Let the tangent at B meet OA in T . It is clear from the figure that the area of $\triangle OAB$ the area of sector $OAB <$ the area of $\triangle OTB$.

The area of $\triangle OAB = \frac{1}{2} OA \cdot BM = \frac{1}{2} a \cdot a \sin \theta = \frac{1}{2} a^2 \sin \theta$.

The area of sector $OAB = \frac{1}{2} a^2 \theta$.

The area of $\triangle OTB = \frac{1}{2} OB \cdot BT = \frac{1}{2} a \cdot a \tan \theta = \frac{1}{2} a^2 \tan \theta$.

$\therefore \frac{1}{2} a^2 \sin \theta < \frac{1}{2} a^2 \theta < \frac{1}{2} a^2 \tan \theta$.

i.e., $\sin \theta < \theta < \tan \theta$.

Dividing by $\sin \theta$, which is positive,

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Let $\theta \rightarrow 0$.

Then $\cos \theta \rightarrow 1$, $\therefore \frac{1}{\cos \theta} \rightarrow 1$

$\therefore \frac{\theta}{\sin \theta}$ lies between the constant 1 and another quantity which $\rightarrow 1$ as $\theta \rightarrow 0$

$$\text{i.e., } \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1$$

$$\text{i.e., } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

Note: The above result is true only if θ is expressed in radian measure.

Exercises

1 Show that $\lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} = 3a^2$, from first principles.

2 Show that $\lim_{x \rightarrow a} \frac{x^4 - a^4}{x - a} = 4a^3$ from first principles.

3 Show that $\lim_{x \rightarrow a} \frac{x - a}{\sqrt{x} - \sqrt{a}} = 2\sqrt{a}$.

4 Show that $\lim_{\theta \rightarrow 0} \frac{\sin m\theta}{\theta} = m$.

6 Find $\lim_{\theta \rightarrow 0} \frac{\sin p\theta}{\sin q\theta}$

Ans. $\frac{p}{q}$

7 Prove that $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$, when θ is negative.

8 Find $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x}$, if x is measured in degrees.

[Hint: Convert x° into radian measure, using the result $\pi \text{ rad} = 180^\circ$]

Ans. $\frac{\pi}{180}$

9 From the inequality $\sin \theta < \theta < \tan \theta$, prove that

$$\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1.$$

10 Evaluate the following:—

1) $\lim_{x \rightarrow 0} \frac{4x^4 - 3x^2 + 7x - 1}{3x^3 - 2x + 5}$

Ans. $-\frac{1}{5}$

2) $\lim_{x \rightarrow 0} \frac{2x + 3x^2 - 4x^3}{3x - 4x^2 + 7x^3}$

Ans. $\frac{2}{3}$

3) $\lim_{x \rightarrow 1} \frac{x^2 - 7x + 6}{2x^2 - 3x + 1}$

Ans. $-\frac{5}{2}$

4) $\lim_{x \rightarrow 2} \frac{2x^2 - 7x + 6}{3x^2 - 7x + 2}$

Ans. $\frac{1}{5}$

5) $\lim_{x \rightarrow -1} \frac{3x^2 + 4x + 1}{4x^2 + 3x - 1}$

Ans. $\frac{2}{5}$

6) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Ans. 1

7) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

Ans. 0

8) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

Ans. $\frac{1}{2}$

9) $\lim_{x \rightarrow 0} \frac{1 - \cos px}{1 - \cos qx}$

Ans. $\frac{p^2}{q^2}$

CHAPTER III

DIFFERENTIATION

9 Differential coefficient If y is a given function of x , very often it is useful to find out how the function changes in value, when the independent variable x is changed. Let x change by a quantity h and let the corresponding change in y be k , so that $y+k=f(x+h)$.

$$\text{i.e., } k = f(x+h) - f(x)$$

$$\therefore \frac{k}{h} = \frac{f(x+h) - f(x)}{h}$$

The ratio of the change in the function to the change in the variable is thus given by $\frac{f(x+h) - f(x)}{h}$.

If now we make the change in x very small, $\frac{f(x+h) - f(x)}{h}$

may tend to a limit. If this limit exists, it is called the **differential coefficient** of y with respect to x . By some writers it is also called the **derivative** of y with respect to x .

This idea of a differential coefficient i.e., the limit of the ratio of the change in the function to the change in the variable when the change in the variable is made to tend to zero was conceived at about the same time by Newton of England and Leibnitz of Germany. Beginning from this small idea, this branch of Mathematics known as Calculus has advanced so much that a good understanding of the subject is absolutely essential to any student of Modern Mathematics.

The change in x is generally called the increment and is conveniently represented by the symbol δ , i.e., δx represents a small change in x , δy represents a small change in y and so on. These are used instead of the h and k used above, as they explain themselves as increments in x and y respectively.

With this notation, the differential coefficient or the derivative of y with respect to $x = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$, when that limit exists. A convenient notation for this limit is $\frac{dy}{dx}$ where $\frac{dy}{dx}$ means the limit of $\frac{\delta y}{\delta x}$ as $\delta x \rightarrow 0$. dy and dx have no separate existence, though δy and δx have. $\frac{dy}{dx}$ represents only a brief and convenient way of writing 'the differential coefficient of y with respect to x '. $\frac{dy}{dx}$ is not a quotient but the limit of the quotient $\frac{\delta y}{\delta x}$. The student must note the importance of the notation which is so very elegant, brief and useful. Another notation for $\frac{dy}{dx}$ is $f'(x)$.

We shall now illustrate the idea of the differential coefficient by working a few examples.

Worked Examples

1 Let $y = k$, a constant.

Let x change by δx and let the corresponding change in y be δy .

$$\text{i. e. } y + \delta y = k$$

$$\text{i. e. } \delta y = 0$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{0}{\delta x} = 0.$$

Thus the differential coefficient of a constant is 0.

2 Let $y = x$.

Let x change by δx and let the corresponding change in y be δy .

$$\text{Then } y + \delta y = x + \delta x$$

$$\text{i. e. } \delta y = \delta x$$

$$\text{i. e. } \frac{\delta y}{\delta x} = 1$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 1.$$

Thus the differential coefficient of x with respect to x is 1

3 Let $y = x^2$

With the usual notation,

$$y + \delta y = (x + \delta x)^2$$

$$\therefore \delta y = 2x\delta x + (\delta x)^2$$

$$\therefore \frac{\delta y}{\delta x} = 2x + \delta x$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 2x.$$

$\therefore$ the differential coefficient of x^2 is $2x$.

10 If n is a rational number, and if $y = x^n$, show that

$$\frac{dy}{dx} = nx^{n-1}.$$

$y = x^n$ where n is a rational number

$$\text{Then } y + \delta y = (x + \delta x)^n$$

$$\therefore \delta y = (x + \delta x)^n - x^n$$

$$\therefore \frac{\delta y}{\delta x} = \frac{(x + \delta x)^n - x^n}{\delta x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{(x + \delta x)^n - x^n}{\delta x} \\ &= \lim_{x + \delta x \rightarrow x} \frac{(x + \delta x)^n - x^n}{(x + \delta x) - x} \\ &= nx^{n-1}, \text{ from Art. 7.} \end{aligned}$$

Thus the differential coefficient of x^n is nx^{n-1} when n is any rational number.

Note: The functions in worked examples Art. 9 are special cases of x^n .

11 If $y = kf(x)$, where k is a constant, show that

$$\frac{dy}{dx} = kf'(x).$$

$$y = kf(x)$$

then $y + \delta y = kf(x + \delta x)$, $\therefore k$ is a constant

$$\therefore \delta y = k[f(x + \delta x) - f(x)]$$

$$\therefore \frac{\delta y}{\delta x} = \frac{k[f(x + \delta x) - f(x)]}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} k \frac{[f(x + \delta x) - f(x)]}{\delta x} = kf'(x).$$

12 If $y = u + v - w$, where u, v and w are functions of x ,

then show that $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$.

$$y = u + v - w.$$

Let x change by δx and let the corresponding changes in u, v, w and y be $\delta u, \delta v, \delta w$ and δy .

$$\therefore y + \delta y = (u + \delta u) + (v + \delta v) - (w + \delta w)$$

$$\therefore \delta y = \delta u + \delta v - \delta w.$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} - \frac{\delta w}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} - \frac{\delta w}{\delta x} \right) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

Note: It follows that the differential coefficient of the algebraic sum of any finite number of functions of x is the algebraic sum of their differential coefficients.

Exercises

Differentiate with respect to x :-

1) x^{10} . 2) $x^{\frac{3}{2}}$. 3) $x^{\frac{9}{4}}$. 4) $\frac{1}{\sqrt{x}}$. 5) $\frac{4}{x^2}$. 6) $\frac{3}{x^2}$.

7) $2x^3 - \frac{3}{\sqrt{x}}$. 8) $\frac{4}{x} - \frac{5}{x^2}$. 9) $\frac{1}{x} + \frac{2}{x^2} - \frac{3}{x^3}$. 10) $ax^m + \frac{b}{x^n}$.

11) $\frac{ax^2 + bx + c}{x}$. 12) $\frac{ax^2 + bx + c}{x^3}$.

Ans. 1) $10x^9$. 2) $\frac{3}{2\sqrt{x}}$. 3) $\frac{9}{4x^{\frac{5}{2}}}$. 4) $-\frac{1}{2x^{\frac{3}{2}}}$.

5) $-\frac{16}{x^5}$. 6) $-\frac{6}{x^3}$. 7) $6x^2 + \frac{3}{2x^{\frac{3}{2}}}$. 8) $-\frac{4}{x^2} + \frac{10}{x^3}$.

9) $-\frac{1}{x^2} - \frac{4}{x^3} + \frac{9}{x^4}$. 10) $amx^{m-1} - \frac{bn}{x^{n+1}}$. 11) $a - \frac{c}{x^2}$.

12) $-\frac{a}{x^2} - \frac{2b}{x^3} - \frac{3c}{x^4}$.

13 If $y = uv$, where u and v are both functions of x , then show that $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$.

Let x change by δx . Let the corresponding changes in u, v and y be $\delta u, \delta v$ and δy respectively.

$$y + \delta y = (u + \delta u)(v + \delta v)$$

$$\therefore \delta y = u\delta v + v\delta u + \delta u \cdot \delta v$$

$$\therefore \frac{\delta y}{\delta x} = u \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} + \delta u \cdot \frac{\delta v}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \left(u \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} + \delta u \cdot \frac{\delta v}{\delta x} \right)$$

$$= u \frac{dv}{dx} + v \frac{du}{dx} + 0 \cdot \frac{dv}{dx}$$

$$= u \frac{dv}{dx} + v \frac{du}{dx}.$$

14 If $y = \frac{u}{v}$, show that $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$.

Let x change by δx and let the changes in u, v and y be $\delta u, \delta v$ and δy respectively.

$$\begin{aligned}
 y + \delta y &= \frac{u + \delta u}{v + \delta v} \\
 \therefore \delta y &= \frac{u + \delta u}{v + \delta v} - \frac{u}{v} \\
 &= \frac{(u + \delta u)v - (v + \delta v)u}{v(v + \delta v)} \\
 &= \frac{v\delta u - u\delta v}{v(v + \delta v)} \\
 \therefore \frac{\delta y}{\delta x} &= \frac{v \frac{\delta u}{\delta x} - u \frac{\delta v}{\delta x}}{v(v + \delta v)} \\
 \therefore \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{v \frac{\delta u}{\delta x} - u \frac{\delta v}{\delta x}}{v(v + \delta v)} \\
 &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v(v+0)} \\
 &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
 \end{aligned}$$

15 Composite Function or Function of a Function

If $y = f(u)$ and $u = F(x)$, to prove that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Let x change by δx and let the corresponding changes in u, y be $\delta u, \delta y$ respectively.

It is clear that when $\delta x \rightarrow 0$, δu and δy must also tend to 0, for otherwise $\frac{du}{dx}$ and $\frac{dy}{du}$ do not exist.

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x} \right) \\
 &= \lim_{\delta u \rightarrow 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \rightarrow 0} \frac{\delta u}{\delta x} \\
 &= \frac{dy}{du} \cdot \frac{du}{dx}
 \end{aligned}$$

This is known as the chain Rule.

Note: 1) The student must carefully note that the R.H.S. is a product of two limits $\lim_{\delta u \rightarrow 0} \frac{\delta y}{\delta u}$ and $\lim_{\delta x \rightarrow 0} \frac{\delta u}{\delta x}$. He must guard himself against supposing that du can be cancelled, for as explained earlier du is not a factor either in the numerator of $\frac{du}{dx}$ nor in the denominator of $\frac{dy}{du}$.

2) Similarly $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$.

16 If $y = \sin x$, prove that $\frac{dy}{dx} = \cos x$.

Let x take an increment δx and let the corresponding increment in y be δy .

$$\text{Then } y + \delta y = \sin(x + \delta x)$$

$$\begin{aligned}
 \therefore \delta y &= \sin(x + \delta x) - \sin x \\
 &= 2 \cos \frac{2x + \delta x}{2} \sin \frac{\delta x}{2}
 \end{aligned}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{2 \cos \frac{2x + \delta x}{2} \sin \frac{\delta x}{2}}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{2 \cos \frac{2x + \delta x}{2} \sin \frac{\delta x}{2}}{\delta x}$$

$$\begin{aligned}
 &= \lim_{\delta x \rightarrow 0} \frac{2 \cos \frac{2x + \delta x}{2} \sin \frac{\delta x}{2}}{\frac{\delta x}{2} \times 2} \\
 &= \cos x \quad \because \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \rightarrow 1, \text{ by Art. 8.}
 \end{aligned}$$

$\therefore$ the differential coefficient of $\sin x$ is $\cos x$.

17 If $y = \cos x$, prove that $\frac{dy}{dx} = -\sin x$.

Let x change by δx and let the corresponding change in y be δy .

$$y + \delta y = \cos(x + \delta x)$$

$$\therefore \delta y = \cos(x + \delta x) - \cos x$$

$$= -2 \sin \frac{2x + \delta x}{2} \sin \frac{\delta x}{2}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{-2 \sin \frac{2x + \delta x}{2} \sin \frac{\delta x}{2}}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{-2 \sin \frac{2x + \delta x}{2} \sin \frac{\delta x}{2}}{\frac{\delta x}{2} \times 2}$$

$$= -\sin x \quad \because \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \rightarrow 1, \text{ by Art. 8.}$$

18 If $y = \tan x$, show that $\frac{dy}{dx} = \sec^2 x$.

Let x change by δx and let the corresponding change in y be δy .

$$\text{Then } y + \delta y = \tan(x + \delta x)$$

$$\therefore \delta y = \tan(x + \delta x) - \tan x$$

$$= \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \delta x) \cos x - \cos(x + \delta x) \sin x}{\cos(x + \delta x) \cos x}$$

$$= \frac{\sin \delta x}{\cos(x + \delta x) \cos x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \left(\frac{\sin \delta x}{\delta x} \right) \frac{1}{\cos(x + \delta x) \cos x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x. \quad \left(\frac{\sin \delta x}{\delta x} \rightarrow 1, \text{ Art. 8.} \right)$$

Note: We can similarly find from definition the derivatives of $\cot x$, $\sec x$, and $\operatorname{cosec} x$ with respect to x .

We find that 1) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$2) \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$3) \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$$

Worked Examples

1 Differentiate $(3x - 1)^7$ with respect to x .

Let $u = 3x - 1$, so that $y = u^7$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 7u^6 \cdot 3$$

$$= 21(3x - 1)^6.$$

2 Differentiate $(1 - 3x - x^2)^{\frac{1}{4}}$ with respect to x .

Let $u = 1 - 3x - x^2$

Then $y = u^{\frac{1}{4}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{4} u^{-\frac{3}{4}} \cdot (-3 - 2x)$$

$$= \frac{-(3 + 2x)}{4(1 - 3x - x^2)^{\frac{3}{4}}}$$

3 If $y = (3x^2 + 1)(2x - 1)^7$, find $\frac{dy}{dx}$.

Let $u = 3x^2 + 1$ and $v = (2x - 1)^7$.

Then $\frac{du}{dx} = 6x$ and $\frac{dv}{dx} = 7(2x - 1)^6 \cdot 2$

$$= 14(2x - 1)^6$$

$$\therefore \frac{dy}{dx} = (3x^2 + 1) 14(2x - 1)^6 + (2x - 1)^7 6x, \text{ by Art. 13,}$$

$$= 2(2x - 1)^6 (27x^2 - 3x + 7).$$

4 If $y = \frac{x-1}{\sqrt{3x+1}}$, find $\frac{dy}{dx}$.

Let $u = x-1$ and $v = \sqrt{3x+1}$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{2\sqrt{3x+1}} \times 3$

$\therefore \frac{dy}{dx} = \frac{\sqrt{3x+1} \cdot 1 - (x-1) \cdot \frac{3}{2\sqrt{3x+1}}}{3x+1}$, by Art. 14
 $= \frac{2(3x+1) - 3(x-1)}{2(3x+1)^{\frac{3}{2}}}$
 $= \frac{3x+5}{2(3x+1)^{\frac{3}{2}}}$

5 If $y = x^2 \sin x$, find $\frac{dy}{dx}$

Let $u = x^2$ and $v = \sin x$

Then $\frac{dy}{dx} = x^2 \cos x + \sin x \cdot 2x$
 $= x^2 \cos x + 2x \sin x$

6 If $y = \frac{3x^2-1}{\cos x}$, find $\frac{dy}{dx}$

Let $u = 3x^2-1$ and $v = \cos x$

$\frac{dy}{dx} = \frac{\cos x(6x) - (3x^2-1)(-\sin x)}{\cos^2 x}$
 $= \frac{6x \cos x + (3x^2-1) \sin x}{\cos^2 x}$

7 Differentiate with respect to x :—

a) $\sin^2 x$

b) $\sin x^2$

a) Let $u = \sin x$ so that $y = u^2$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot \cos x$

$= 2 \sin x \cos x = \sin 2x$

b) Let $u = x^2$ so that $y = \sin u$.

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot 2x$
 $= 2x \cos x^2$

8 If $y = \sin mx$, show that $\frac{dy}{dx} = m \cos mx$.

Let $u = mx$, so that $y = \sin u$.

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
 $= \cos u \cdot m$
 $= m \cos mx$

9 If $y = \cos nx$, show that $\frac{dy}{dx} = -n \sin nx$.

Let $u = nx$, so that $y = \cos u$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
 $= -\sin u \cdot n$
 $= -n \sin nx$

10 If $y = \sin^3 2x$, find $\frac{dy}{dx}$.

$y = (\sin 2x)^3$

Let $v = 2x$, and $u = \sin v$, so that $y = u^3$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$
 $= 3u^2 \cdot \cos v \cdot 2$
 $= 3 \sin^2 2x \cdot \cos 2x \cdot 2$
 $= 6 \sin^2 2x \cdot \cos 2x$

Exercises

1 Find the differential coefficients of the following with respect to x :—

✓ 1) $x^2 \cos 3x$

✓ 2) $\sqrt{16-x^2}$

✓ 3) $\frac{1}{\sqrt{9-2x^2}}$

✓ 4) $\frac{\sin 2x}{\cos 3x}$

✓ 5) $(3-5x^2)^{10}$

✓ 6) $\sin(1-3x)$

- 7) $\sin \frac{1}{x}$ 8) $\sin 2x \cos 3x$ 9) $\cos \sqrt{x}$
 10) $(x^2 - 4x + 1)^3$ 11) $(x^2 + x + 1)^5$ 12) $x(2-x)^4$
 13) $\tan^2(x^2 + 1)$ 14) $\sqrt{\cot x}$ 15) $\sec^3(3x + 5)$
 16) $\operatorname{cosec}^3 x^2$ 17) $\sin \sqrt{x}$ 18) $\tan \sqrt{x+1}$
 19) $\sqrt{\frac{1 - \cos x}{1 + \cos x}}$ 20) $\frac{1 - \sin x}{1 + \sin x}$ 21) $\frac{x^2}{x^3 + 1}$
 22) $\frac{x}{\sqrt{1-x}}$ 23) $\cos \frac{2}{x^2}$ 24) $\tan \frac{x+1}{x+2}$
 25) $\sec(\sin x)$ 26) $\frac{\sin x}{x}$ 27) $\frac{x}{x^2 + a^2}$
 28) $\frac{x^2 + 1}{x^2 - 1}$ 29) $\frac{x^2}{\sqrt{4x+1}}$ 30) $\frac{x}{4-3x}$
- Ans. 1) $2x \cos 3x - 3x^2 \sin 3x$ 2) $\frac{-x}{\sqrt{16-x^2}}$
 3) $\frac{2x}{(9-2x^2)^{\frac{3}{2}}}$ 4) $\frac{2 \cos 2x \cos 3x + 3 \sin 2x \sin 3x}{\cos^2 3x}$
 5) $-100x(3-5x^2)^9$ 6) $-3 \cos(1-3x)$ 7) $-\frac{1}{x^2} \cos \frac{1}{x}$
 8) $2 \cos 2x \cos 3x - 3 \sin 2x \sin 3x$ 9) $-\frac{\sin \sqrt{x}}{2\sqrt{x}}$
 10) $6(x-2)(x^2-4x+1)^2$ 11) $5(2x+1)(x^2+x+1)^4$
 12) $(2-x)^3(2-5x)$
 13) $4x \tan(x^2+1) \sec^2(x^2+1)$ 14) $\frac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$
 15) $9 \sec^3(3x+5) \tan(3x+5)$ 16) $-6x \operatorname{cosec}^3 x^2 \cot x^2$
 17) $\frac{\cos \sqrt{x}}{2\sqrt{x}}$ 18) $\frac{\sec^2 \sqrt{x+1}}{2\sqrt{x+1}}$ 19) $\frac{1}{2} \sec^2 \frac{x}{2}$
 20) $\frac{-2 \cos x}{(1+\sin x)^2}$ 21) $\frac{2x-x^4}{(x^3+1)^2}$ 22) $\frac{2-x}{2(1-x)^{\frac{3}{2}}}$
 23) $\frac{4}{x^3} \sin \frac{2}{x^2}$ 24) $\frac{1}{(x+2)^2} \sec^2 \frac{x+1}{x+2}$
 25) $\cos x \sec(\sin x) \tan(\sin x)$ 26) $\frac{x \cos x - \sin x}{x^2}$

27) $\frac{a^2 - x^2}{(x^2 + a^2)^2}$ 28) $\frac{-4x}{(x^2 - 1)^2}$ 29) $\frac{6x^2 + 2x}{(4x+1)^{\frac{3}{2}}}$
 30) $\frac{4}{(4-3x)^2}$

If $y = uvw$, where u , v and w are functions of x , prove that $\frac{dy}{dx} = vw \frac{du}{dx} + wu \frac{dv}{dx} + uv \frac{dw}{dx}$.

3 Prove that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

4. If $y = (x+2)(2x-1)(3x+7)$, find $\frac{dy}{dx}$.

Ans. $18x^2 + 46x + 15$

5 Find, from first principles, the differential coefficients of $\cot x$, $\sec x$ and $\operatorname{cosec} x$.

Ans. $-\operatorname{cosec}^2 x$; $\sec x \tan x$; $-\operatorname{cosec} x \cot x$

19 Implicit functions

Most of the functions we have considered so far have been explicitly defined by an algebraic equation in the form $y = f(x)$, e.g. $y = x^3 + x$. In the equation $y^3 + y = x^2 + 1$, it is not easy to solve for y in terms of x , but y is clearly a function of x . Such a function is said to be defined implicitly by the given equation. In general, if $f(x, y) = 0$, then y is called an implicit function of x .

If the implicit form of a function is given, the derivative of y with respect to x can be found out by differentiating each term of the equation in the implicit form and then solving for $\frac{dy}{dx}$.

Worked Example

Find $\frac{dy}{dx}$ if $y^3 + y = x^2 + 1$.

Differentiating each term of the given implicit function with respect to x ,

$$3y^2 \frac{dy}{dx} + \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} (3y^2 + 1) = 2x$$

$$\therefore \frac{dy}{dx} = \frac{2x}{3y^2 + 1}$$

Exercises

Find $\frac{dy}{dx}$ if

- (1) $x^2 + y^2 = a^2$ (2) $x^2 + 3xy + y^2 = 7$ (3) $x + \sqrt{xy} = x^2$
 (4) $\sin y = 2 \cos x$ (5) $x^2 y^2 + y = x^3$ (6) $ax^2 + 2hxy + by^2 = 1$
 (7) $x \sin y + y \sin x = 10$ (8) $x^3 + y^3 = 3xy$

Ans. (1) $-\frac{x}{y}$ (2) $-\frac{2x+3y}{3x+2y}$ (3) $\frac{2\sqrt{xy}(2x-1)-y}{x}$
 (4) $-\frac{2 \sin x}{\cos y}$ (5) $\frac{3x^2 - 2xy^2}{2x^2 y + 1}$ (6) $-\frac{ax+by}{hx+by}$
 (7) $-\frac{\sin y + y \cos x}{x \cos y + \sin x}$ (8) $\frac{x^2 - y}{x - y^2}$

20 Inverse functions

We have seen in chapter I that if $f(x)$ is an one-one and onto function its inverse is given by $f^{-1}(x)$. If $y = f^{-1}(x)$ then $x = f(y)$.

Differentiating with respect to x by chain rule, we get

$$\frac{d}{dy} f(y) \frac{dy}{dx} = 1$$

$$\text{i.e. } \frac{dy}{dx} = \frac{1}{\frac{d}{dy} f(y)} = \frac{1}{f'(y)}$$

21 To find the derivative of $\sin^{-1}x$

$y = \sin^{-1}x$ is defined in $-1 \leq x \leq 1$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

$$\therefore \sin y = x$$

$$\cos y \frac{dy}{dx} = 1, \text{ i.e. } \frac{dy}{dx} = \frac{1}{\cos y}$$

$\cos y = 0$ when $y = \pm \frac{\pi}{2}$ i.e. when $x = \pm 1$

Excluding these values of x ,

$$\frac{dy}{dx} = \frac{1}{\cos y} \text{ when } -\frac{\pi}{2} < y < \frac{\pi}{2} \text{ and } -1 < x < 1$$

$$= + \frac{1}{\sqrt{1-\sin^2 y}} \because \cos y > 0 \text{ for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$= \frac{1}{\sqrt{1-x^2}} \text{ for } -1 < x < 1.$$

22 To find the derivative of $\cos^{-1}x$

$y = \cos^{-1}x$ is defined for $-1 \leq x \leq 1$ and for $0 \leq y \leq \pi$.

$$\cos y = x$$

$-\sin y \frac{dy}{dx} = 1$. Excluding the values 0 and π for y ,

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} (\sin y > 0)$$

$$= -\frac{1}{\sqrt{1-x^2}} (-1 < x < 1)$$

23 To find the derivative of $\tan^{-1}x$

$y = \tan^{-1}x$ is defined for all values of x and for

$$-\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\therefore \tan y = x$$

$\therefore \sec^2 y \frac{dy}{dx} = 1$. Since $\sec^2 y$ does not vanish for

$$-\frac{\pi}{2} < y < \frac{\pi}{2},$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\therefore \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \text{ for all values of } x.$$

Note: The derivatives of $\cot^{-1}x$, $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ can all be derived similarly. The results are as follows:

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2} \text{ for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}} \text{ for } |x| > 1$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}} \text{ for } |x| > 1.$$

Worked Examples

1 If $y = \sin^{-1}x$ find $\frac{dy}{dx}$ from definition.

$$(-1 < x < 1 \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}).$$

Let the increment in x be δx and the corresponding increment in y be δy .

$$\text{Then } y + \delta y = \sin^{-1}(x + \delta x)$$

$$\therefore \sin(y + \delta y) = x + \delta x$$

$$\text{and } \sin y = x$$

$$\therefore \sin(y + \delta y) - \sin y = \delta x$$

$$\therefore 2 \cos\left(y + \frac{\delta y}{2}\right) \sin \frac{\delta y}{2} = \delta x$$

$$\text{i.e., } \frac{\cos\left(y + \frac{\delta y}{2}\right) \sin \frac{\delta y}{2}}{\frac{\delta y}{2}} = \frac{\delta x}{\delta y}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{1}{\cos\left(y + \frac{\delta y}{2}\right) \sin \frac{\delta y}{2}}$$

Let $\delta x \rightarrow 0$, then δy also $\rightarrow 0$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta y \rightarrow 0} \frac{1}{\cos\left(y + \frac{\delta y}{2}\right) \sin \frac{\delta y}{2}} \\ &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} \quad (\cos y > 0) = \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

2 Find $\frac{dy}{dx}$ if $y = \sin^{-1}x^2$.

Let $x^2 = u$ so that $y = \sin^{-1}u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}.$$

3 Find $\frac{dy}{dx}$ if $y = \cos^{-1}(1-2x)$.

Let $1-2x = u$ so that $y = \cos^{-1}u$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{-1}{\sqrt{1-u^2}} \cdot (-2) \\ &= \frac{2}{\sqrt{1-(1-2x)^2}} = \frac{2}{\sqrt{4x-4x^2}} = \frac{1}{\sqrt{x-x^2}}. \end{aligned}$$

4 Find $\frac{dy}{dx}$ if $y = \tan^{-1}(x+1)^2$.

Let $u = (x+1)^2$ so that $y = \tan^{-1}u$

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot 2(x+1) \\ &= \frac{2(x+1)}{1+(x+1)^4} \end{aligned}$$

5 Find $\frac{dy}{dx}$ if $y = \sec^{-1}x^3$.

Let $u = x^3$ so that $y = \sec^{-1}u$.

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u\sqrt{u^2-1}} \cdot 3x^2 \\ &= \frac{3x^2}{x^3\sqrt{x^6-1}} = \frac{3}{x\sqrt{x^6-1}}. \end{aligned}$$

Exercises

Find $\frac{dy}{dx}$ if

1) $y = x^2 \sin^{-1} x$ 2) $y = \cos^{-1} (1-x)^2$ 3) $y = x \tan^{-1} x$

4) $y = (x^2 + 1) \sec^{-1} x$ 5) $y = \tan^{-1} \sqrt{x}$

6) $y = \tan^{-1} \frac{2x}{1-x^2}$ 7) $y = \sin^{-1} \frac{1-x}{x}$

8) $y = \sec^{-1} \sqrt{1+x^2}$ 9) $y = \tan^{-1} \frac{1+x}{1-x}$

Ans. 1) $\frac{x^2}{\sqrt{1-x^2}} + 2x \sin^{-1} x$ 2) $\frac{2(1-x)}{\sqrt{1-(1-x)^2}}$

3) $\frac{x}{1+x^2} + \tan^{-1} x$ 4) $\frac{x^2+1}{x\sqrt{x^2-1}} + 2x \sec^{-1} x$

5) $\frac{1}{2\sqrt{x(1+x)}}$ 6) $\frac{2}{1+x^2}$ 7) $\frac{-1}{x\sqrt{2x-1}}$

8) $\frac{1}{1+x^2}$ 9) $\frac{1}{1+x^2}$

24 Parameters

Sometimes x and y are expressed in terms of a third variable ' t ' which is called a parameter, e.g. $x = f(t)$, $y = \phi(t)$.

Then $t = f^{-1}(x) = g(x)$, say

i.e. $y = \phi(t) = \phi\{(g)x\} = \phi \circ g$.

Hence y is a composite function of x and its derivative with respect to x can be found out in terms of the parameter t . This is known as parametric differentiation.

By the rule of differentiation of a function of a function,

$$\text{we have } \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\phi'(t)}{f'(t)} \quad [f'(t) \neq 0].$$

Worked Examples

1) Find $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$.

$$\frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

2) Find $\frac{dy}{dx}$ if $x = a \cos \theta$, $y = a \sin \theta$.

$$\frac{dx}{d\theta} = -a \sin \theta \text{ and } \frac{dy}{d\theta} = a \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta.$$

Exercises

Find $\frac{dy}{dx}$ if

1) $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

Ans. $-\tan \theta$

2) $x = a \sec \theta$, $y = b \tan \theta$.

Ans. $\frac{b}{a} \operatorname{cosec} \theta$

3) $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$

Ans. $\cot \frac{\theta}{2}$

4) $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$

Ans. $\frac{2t-t^4}{1-2t^3}$

5) $x = 2 \sin \theta$, $y = \cos 2\theta$

Ans. $-2 \sin \theta$

6) $x = 3 \cos \theta - \cos^3 \theta$,
 $y = 3 \sin \theta - \sin^3 \theta$

Ans. $-\cot^3 \theta$

7) $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

Ans. $\frac{t^2-1}{2t}$

8) $x = a \cos^4 \theta$, $y = a \sin^4 \theta$

Ans. $-\tan^2 \theta$

TANGENTS AND NORMALS

25 Geometrical meaning of the derivative

Let $P(x, y)$ be a point on a continuous curve whose equation is $y = f(x)$.

Let Q be an adjacent point on the curve, whose co-ordinates are $(x + \delta x, y + \delta y)$.

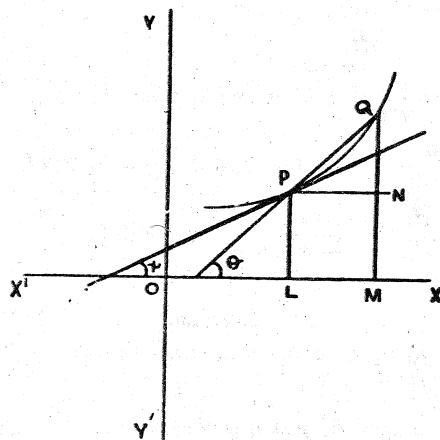


Fig. 10

Draw PL, QM perpendicular to OX and PN perpendicular to QM . Join PQ and let it make an angle θ with OX .

Then $PN = LM = OM - OL = x + \delta x - x = \delta x$.

$NQ = MQ - MN = MQ - LP = y + \delta y - y = \delta y$.

Since PN is $\parallel$ to OX , $\angle NPQ = \theta$.

$$\therefore \tan \theta = \tan \angle NPQ = \frac{NQ}{PN} = \frac{\delta y}{\delta x}.$$

Let Q approach P along the curve. In the limiting position the chord QP becomes the tangent to the curve at P . Let the tangent at P make an angle ψ with OX .

Let $\delta x \rightarrow 0$. Then $M \rightarrow L$, i.e. $Q \rightarrow P$ along the curve.
 $\therefore$ the chord QP tends to become the tangent at P ,
 i.e. $\tan \theta \rightarrow \tan \psi$

$$\therefore \text{in the limit, } \tan \psi = \lim \tan \theta = \lim \frac{\delta y}{\delta x} = \frac{dy}{dx}.$$

Hence $\frac{dy}{dx}$ gives the value of the slope of the tangent at $P(x, y)$ or the slope of the curve at (x, y) .

Note 1. If $\frac{dy}{dx} = 0$, at a point, $\tan \psi = 0$, i.e. the tangent at that point is parallel to the X -axis.

2. If $\frac{dy}{dx} = \infty$, then $\psi = 90^\circ$, i.e. the tangent is $\parallel$ to the Y -axis.

3. The slope of the curve at a point $P(x_1, y_1)$ is denoted by $\left(\frac{dy}{dx}\right)_P$, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ or $f'(x_1)$.

26 Equation of the tangent to a curve $y = f(x)$ at the point (x_1, y_1) on it

We know that the equation of the straight line through (x_1, y_1) and having a slope ' m ' is $y - y_1 = m(x - x_1)$

We have seen that the slope of the tangent at (x_1, y_1) on $y = f(x)$ is $f'(x_1)$.

$$\therefore \text{the equation required is } y - y_1 = f'(x_1)(x - x_1).$$

27 Equation of the normal to a curve $y = f(x)$ at the point (x_1, y_1) on it

The normal to a curve at the point $P(x_1, y_1)$ on it is the line through the point P perpendicular to the tangent at the point. If the slope of the tangent is ' m ' we know

$$\text{that the slope of the normal is } -\frac{1}{m}.$$

Hence the normal is the line through (x_1, y_1) and having a slope $= -\frac{1}{m}$.

$$\therefore \text{its equation is } y - y_1 = -\frac{1}{f'(x_1)}(x - x_1).$$

Worked Examples

1 Find the equations of the tangent and normal to the curve $y = x^2$ at the point $(1, 1)$ on it.

Equation to the given curve is $y = x^2$

$$\therefore \frac{dy}{dx} = 2x$$

$$\therefore \text{at } (1, 1) \frac{dy}{dx} = 2$$

$\therefore$ the equation of the tangent is

$$y - 1 = 2(x - 1) \text{ or } 2x - y - 1 = 0.$$

The slope of the normal is $-\frac{1}{2}$

$\therefore$ the equation of the normal is $y - 1 = -\frac{1}{2}(x - 1)$
i.e. $2y + x - 3 = 0$.

2 Find the equations of the tangent and normal at the point $(1, 4)$ on $xy = 4$.

$$xy = 4$$

$$\therefore y = \frac{4}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{4}{x^2}$$

$$\therefore \text{at the point } (1, 4) \frac{dy}{dx} = -4$$

i.e. the slope of the tangent at $(1, 4) = -4$

$\therefore$ the equation of the tangent is $y - 4 = -4(x - 1)$
or $4x + y - 8 = 0$

The slope of the normal at $(1, 4) = \frac{1}{4}$

$\therefore$ the equation of the normal at $(1, 4)$ is $y - 4 = \frac{1}{4}(x - 1)$
or $x - 4y + 15 = 0$.

3 Find the equations of the tangent and normal at $(2, 1)$ to the curve $x^2 + 2y^2 = 6$.

$$x^2 + 2y^2 = 6$$

Differentiating with respect to x , we have

$$2x + 4y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{2y}$$

$$\therefore \text{at the point } (2, 1), \frac{dy}{dx} = -1$$

i.e. the slope of the tangent at $(2, 1)$ is -1

$\therefore$ the equation of the tangent at $(2, 1)$ is $y - 1 = -(x - 2)$
or $x + y - 3 = 0$

The equation of the normal at $(2, 1)$ is $y - 1 = x - 2$
or $x - y - 1 = 0$.

Exercises

1 Find the points on the curve $y = 2x^3 - 3x^2 + 5$ where the tangent is parallel to the X -axis.

Ans. $(0, 5), (1, 4)$

2 Find the equations of the tangent and normal to the curve $4y = x^3$ at the point $(2, 2)$.

Ans. $3x - y - 4 = 0; x + 3y - 8 = 0$

3 Find the equations of the tangent and normal to the curve $y^2 = 4x$ at the point $(1, 2)$.

Ans. $x - y + 1 = 0; x + y - 3 = 0$

4 Find the equation of the tangent to the curve $y = 4x^2$ which is parallel to the straight line $y = 4x$.

Ans. $y = 4x - 1$

5 Find the equation of the tangent to the curve $y^2 = 3x + 2$ which is parallel to the line $3x - 4y + 1 = 0$.

Ans. $3x - 4y + 6 = 0$

6 Find the equation of the normal to the curve $y = 6 - x^2$ which is parallel to the line $x - 4y + 3 = 0$.

Ans. $x - 4y + 6 = 0$

✓ 7 Show that the curves $y = x^3 + x + 1$ and $2y = x^3 + 5x$ touch at the point $(1, 3)$. Find the equation of the common tangent at the point.

Ans. $4x - y - 1 = 0$

✓ 8 Find the equations of the tangent and normal to the curve $(1 + x^2)y = 2 - x$ where it crosses the X -axis.

Ans. $x + 5y - 2 = 0$, $5x - y - 10 = 0$

✓ 9 Find the equations of the tangent and normal to the curve $y = 3x^2 + 2$ where the slope of the tangent is -2 .

Ans. $6x + 3y - 5 = 0$; $x - 2y + 5 = 0$

✓ 10 The tangent to the curve $y = 5 - 2x^2$ at the point $x = 1$ cuts the axes at A and B . Find the area of the triangle AOB .

Ans. $\frac{49}{8}$ sq. units

✓ 11 Find the equations of the tangent and normal at $(2, 2)$ on the curve $3y^2 = x^2(x + 1)$.

Ans. $4x - 3y - 2 = 0$; $3x + 4y - 14 = 0$

✓ 12 Find the equations of the tangent and normal to the curve $x^2 - 3y^2 = 54$ at the point $(9, -3)$.

Ans. $x + y - 6 = 0$; $x - y - 12 = 0$

28 Angle between curves

The angle between two curves is the angle between the tangents to the two curves at a point of intersection of

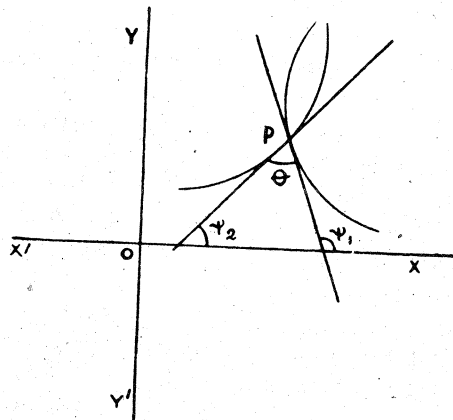


Fig. 11

the curves. If the curves $y = f(x)$ and $y = \phi(x)$ intersect at $P(x_1, y_1)$, and ψ_1, ψ_2 are the inclinations to the X -axis of the tangents to the curves at P , the angle of their intersection is given by $\theta = \psi_1 - \psi_2$.

We know that $\tan \psi_1 = f'(x_1) = m_1$, say

and $\tan \psi_2 = \phi'(x_1) = m_2$, say.

Then $\tan \theta = \tan(\psi_1 - \psi_2)$

$$= \frac{\tan \psi_1 - \tan \psi_2}{1 + \tan \psi_1 \tan \psi_2}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2}$$

Note: 1) If $m_1 = m_2$ the curves touch at P .

2) If $m_1 = -\frac{1}{m_2}$ the curves cut orthogonally at P (or the curves cut at right angles).

3) It is usual to give the acute angle between the curves as the angle between the curves, i.e. if θ is the acute angle between the curves,

$$\text{then } \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Worked Examples

1. Find the angle of intersection of the curves $4y = x^3$ and $y = 6 - x^2$ at the point $(2, 2)$.

For the curve $4y = x^3$, $4 \frac{dy}{dx} = 3x^2$

$$\therefore \frac{dy}{dx} = \frac{3x^2}{4}$$

$$\therefore \text{ at } (2, 2), \frac{dy}{dx} = \frac{3 \times 4}{4} = 3.$$

For the curve, $y = 6 - x^2$, $\frac{dy}{dx} = -2x$

$$\therefore \text{ at } (2, 2), \frac{dy}{dx} = -4.$$

If θ is the required angle, then $\tan \theta = \left| \frac{3+4}{1-12} \right| = \frac{7}{11}$.
 $\therefore \theta = \tan^{-1} \frac{7}{11}$.

- 2 Find the angle of intersection of the curves $y = x^2 - 1$ and $y = -2x^2 + 2$.

To find where the two curves cut, solve between the equations.

$$y = x^2 - 1$$

$$y = -2x^2 + 2$$

Eliminating y , we get $x^2 - 1 = -2x^2 + 2$

$$\therefore 3x^2 = 3$$

$$\therefore x^2 = 1, x = \pm 1 \text{ and } y = 0.$$

$\therefore$ the points of intersection are $(1, 0)$ and $(-1, 0)$

Let us first consider the point $(1, 0)$

For the curve $y = x^2 - 1$, $\frac{dy}{dx} = 2x$

$$\therefore \text{ at } (1, 0), \frac{dy}{dx} = 2$$

For the curve $y = -2x^2 + 2$, $\frac{dy}{dx} = -4x$

$$\therefore \text{ at } (1, 0), \frac{dy}{dx} = -4$$

If θ is the acute angle between the curves at this

$$\text{point, then } \tan \theta = \left| \frac{2+4}{1-8} \right| = \frac{6}{7}$$

$$\therefore \theta = \tan^{-1} \frac{6}{7}.$$

Let us take the second point of intersection $(-1, 0)$

For the curve $y = x^2 - 1$, $\frac{dy}{dx} = 2x$

and for the curve $y = -2x^2 + 2$, $\frac{dy}{dx} = -4x$

If ϕ is the acute angle between the curves at this

$$\text{point then } \tan \phi = \left| \frac{-2-4}{1-8} \right| = \frac{6}{7}$$

$$\therefore \phi = \tan^{-1} \frac{6}{7}.$$

Exercises

1 Show that the curves $y = 2x^2 + 3x$ and $3y = 3x^2 - x$ cut orthogonally at the origin.

2 Find the angle of intersection of the curves $y^2 = 4x$ and $x^2 + y^2 = 5$.
 Ans. $\tan^{-1} 3$

3 Find the angle of intersection of the curves $y^2 = 2x$ and $x^2 + 2y^2 = 14x - 16$ at $(2, 2)$.
 Ans. $\tan^{-1} \frac{6}{3}$

4 Find the angle of intersection of the curves $y^2 = 4x^3$ and $y^2 = 6 - 2x$.
 Ans. $\tan^{-1} 7$

5 Find the value of 'a' such that the curve $y = x^2 - a^2$ may cut the x-axis at an angle of 45° .
 Ans. $\pm \frac{1}{2}$

6 Find the angle of intersection of the curves $x^2 + y^2 = 8$ and $x^2 = 2y$.
 Ans. $\tan^{-1} 3$

29 Subtangents and subnormals

Let the tangent and normal at $P(x_1, y_1)$ on the curve $y = f(x)$ meet the x-axis at T and G respectively. Let

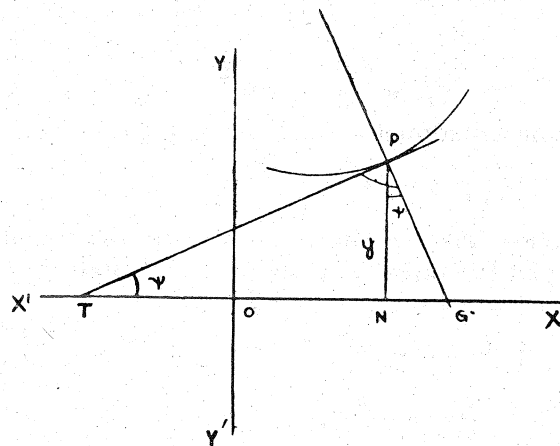


Fig. 12

PN be the ordinate through P . Then TN is called the **subtangent** and NG is called the **subnormal** at P .

If $\angle XTP = \psi$, then from the rightangled triangle TGP have $\angle NPG = \psi$.

$$\text{Also } \tan \psi = \left(\frac{dy}{dx} \right)_P = f'(x_1)$$

$$\therefore \text{ subtangent } TN = \frac{NP}{\tan \psi} = \frac{y_1}{f'(x_1)}.$$

$$\text{Subnormal } NG = NP \tan \psi = y_1 f'(x_1).$$

Worked Examples

1 Find the subtangent and the subnormal at any point on the curve $y^2 = 4ax$.

$$y^2 = 4ax$$

$$\therefore 2y \frac{dy}{dx} = 4a$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \text{ subtangent at } (x_1, y_1) = \frac{y_1}{f'(x_1)} = \frac{y_1}{\left(\frac{2a}{y_1} \right)}$$

$$= \frac{y_1^2}{2a} = \frac{4ax_1}{2a} = 2x_1$$

$$\text{and subnormal at } (x_1, y_1) = y_1 f'(x_1)$$

$$= y_1 \left(\frac{2a}{y_1} \right) = 2a.$$

2 Show that for the curve $x = ay^n$ the subtangent at any point is n times the abscissa of the point.

$$x = ay^n$$

$$\therefore 1 = any^{n-1} \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{nay^{n-1}}$$

$$\therefore \text{ subtangent at } (x_1, y_1) = \frac{y_1}{f'(x_1)} = y_1 nay_1^{n-1}$$

$$= nay_1^n = nx_1.$$

Exercises

1 Find the subtangent and subnormal at $(1, 4)$ on the curve $y = x^3 + x^2 + 2$. Ans. $\frac{4}{3}, 20$

2 Find the subtangent and subnormal at $(1, \frac{1}{2})$ on the curve $y = \frac{x}{x+1}$. Ans. $2, \frac{1}{8}$

3 Prove that for the curve $xy = c^2$ the subnormal varies as the cube of the ordinate.

4 Find the subtangent and subnormal at any point on the curve $y^n = a^{n-1}x$. Ans. $nx_1, \frac{y_1^2}{nx_1}$

5 Find the subtangent and subnormal at any point on the curve $y = b \sin \frac{x}{a}$. Ans. $a \tan \frac{x}{a}; \frac{b^2}{a} \sin \frac{x}{a} \cos \frac{x}{a}$

6 In the curve $x^3y^2 = a^4$, show that the subtangent at any point varies as the abscissa of the point.

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DERIVATIVE AS A RATE MEASURER

30 If y is a given function of x , say $f(x)$, then for any value of x there is a corresponding value of y . Thus when $x=a$, $y=f(a)$ and when $x=b$, $y=f(b)$.

Thus for the change in the value of x from a to b , the function changes from $f(a)$ to $f(b)$.

$\therefore \frac{f(b)-f(a)}{b-a}$ measures the average rate of change in the value of the function for the change in the variable from a to b .

31 **Speed.** We shall illustrate the above definition by an example of a moving point. Let us suppose that a point moves from a fixed point O through a distance s along a straight line given by some relation, say $s=f(t)$, where t is the time that has elapsed after the instant of reckoning.



At time a , let the point be at P where $OP=f(a)$ and at time b , let the point be at Q where $OQ=f(b)$.

Then during the interval $b-a$, the point has moved through a distance $f(b)-f(a)$.

$\therefore$ the average speed of the moving point is $\frac{f(b)-f(a)}{b-a}$.

It does not follow from the above argument that the speed of the point will throughout be $\frac{f(b)-f(a)}{b-a}$. It may be sometimes more, and sometimes less, than this value.

All the same, it is reasonable to argue that if we take an interval smaller than PQ , and consider the average speed during that shorter interval, the result obtained is more likely to represent the actual motion of the point at time a . The extent of inaccuracy involved in taking a

shorter interval commencing with a is evidently less than in taking a longer interval. Further the shorter the interval, the less is the inaccuracy and the more accurate will be the calculated value.

It is at this juncture that the fundamental idea of calculus enters into our calculations.

If the distance $OP=s$, then PQ is a small increment of s and can be conveniently represented by δs traversed by the point in a short interval of time δt .

Thus the average speed during the short interval is $\frac{\delta s}{\delta t}$.

If we now want to find the speed of the point at P , we can say that the speed of the point will be the same as the speed during the interval δt commencing at P when that interval δt is made to tend to zero. Thus the speed of the point is the limit of $\frac{\delta s}{\delta t}$ when $\delta t \rightarrow 0$.

Thus velocity which is defined as rate of change of distance is expressed in the language of the calculus by $\frac{ds}{dt}$. Thus the differential coefficient $\frac{ds}{dt}$ is the rate of change of s with respect to t , i.e. $v = \frac{ds}{dt}$.

32 Acceleration

Let us take the case of the moving point again. Let its velocity at P be v and its velocity at Q be V . Then the change in velocity is $V-v$ during the interval $b-a$.

$\therefore$ the average rate of change of velocity = $\frac{V-v}{b-a}$.

If we now want to find the rate of change of velocity at P i.e., the acceleration at P , we are likely to get a sufficiently approximate value by taking $b-a$ to be small. The smaller we take this interval, the more accurate is the answer likely to be.

If we take the change in velocity to be small, say δv which has taken place in the small interval δt , the average rate of change during the interval δt is $\frac{\delta v}{\delta t}$.

If we want to find the acceleration at P , we can say that it is the same as the average acceleration during an interval δt commencing from P when that interval is made to tend to zero.

$$\text{Thus acceleration at } P = \lim_{\delta t \rightarrow 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}.$$

Thus the acceleration of a moving point defined as the rate of change of velocity is represented by $\frac{dv}{dt}$; in other words, $\frac{dv}{dt}$ is the rate of change of v with respect to t .

In the same way, if y is a function of x , δy denotes the change in y due to a small change δx in x . $\frac{\delta y}{\delta x}$ represents the average rate of change of y with respect to x during the change. If however δx is made to tend to 0, $\frac{\delta y}{\delta x}$ tends to a limit which measures the rate of change of y with respect to x .

Worked Examples

1 A point moves on a straight line so that its distance s from a fixed point on the line at time t is $5 \cos 2t$. Find its velocity and acceleration in terms of s .

$$\begin{aligned} s &= 5 \cos 2t \\ \text{vel. } v &= \frac{ds}{dt} = -5 \sin 2t \times 2 \\ &= -10 \sin 2t \dots (1) \\ &= -10 \sqrt{1 - \frac{s^2}{25}} \\ &= -2\sqrt{25 - s^2}. \end{aligned}$$

$$\begin{aligned} \text{Acceleration} &= \frac{dv}{dt} = -10 \cdot \cos 2t \cdot 2, \text{ from (1)} \\ &= -20 \cos 2t \\ &= -4s. \end{aligned}$$

2 The volume of a sphere is increasing at 2 cu.cnr. per sec. Find the rate of increase of the radius and of the surface area when the volume is 288π cu.cm.

[If r is the radius of a sphere, its volume $= \frac{4}{3}\pi r^3$ and surface area $= 4\pi r^2$].

$$\text{Vol. } V = \frac{4}{3}\pi r^3.$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

$$\text{By hypothesis, } \frac{dV}{dt} = 2$$

$$\therefore \frac{dr}{dt} = \frac{2}{4\pi r^2} = \frac{1}{2\pi r^2}$$

At the instant under consideration Vol $V = \frac{4}{3}\pi r^3 = 288\pi$
i.e., $r^3 = 216$ or $r = 6$

$$\therefore \frac{dr}{dt} = \frac{1}{2\pi \times 36} = \frac{1}{72\pi} \text{ cm. per sec.}$$

$$\text{Area of surface, } A = 4\pi r^2$$

$$\begin{aligned} \frac{dA}{dt} &= 4\pi \cdot 2r \frac{dr}{dt} \\ &= 8\pi \times 6 \times \frac{1}{72\pi} = \frac{1}{3} \text{ sq. cm per sec.} \end{aligned}$$

3 A circular patch of oil spreads on water, the area growing at the rate of 6 sq. inches per minute. How fast is the radius increasing when the radius is 2 inches?

Area $A = \pi r^2$, if r is the radius of the patch.

$$\therefore \frac{dA}{dt} = \pi \cdot 2r \frac{dr}{dt}$$

$$\text{By hypothesis, } \frac{dA}{dt} = 6$$

$$\therefore \frac{dr}{dt} = \frac{6}{2\pi r} = \frac{3}{\pi r}$$

At the moment under consideration $r = 2$

$$\therefore \frac{dr}{dt} \text{ required} = \frac{3}{2\pi} \text{ in. per min.}$$

4 A rod AB 13 ft. long slides with its ends A and B on two straight lines which meet at right angles at O . If the end A slips away from O at the rate of 5 ft. per sec., find the speed of the end B when $OB = 5$.

Let $OA = x$ and $OB = y$

We are given that $\frac{dx}{dt} = 5$.

$$x^2 + y^2 = 169, \text{ from } \triangle AOB.$$

$$y^2 = 169 - x^2$$

$$y = \sqrt{169 - x^2}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{169 - x^2}} (-2x) \frac{dx}{dt}$$

$$= \frac{-x}{\sqrt{169 - x^2}} \frac{dx}{dt}$$

When $y = 5$, $x = 12$.

$$\therefore \frac{dy}{dt} \text{ required} = \frac{-12}{5} \times 5$$

$$= -12 \text{ ft. per sec.}$$

Note: The negative value for $\frac{dy}{dt}$ shows that with the lapse of time, y decreases.

5 A filter paper is in the form of a right circular cone with its altitude = 5 inches and the radius of the base = 3 inches. If water flows out at the constant rate of 3 cu. inches per minute, find the rate at which the level of water falls when the depth of the water is 3 inches.

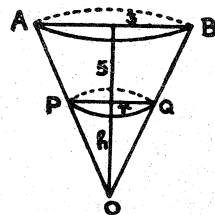


Fig. 13

At any time let h be the depth of the water and r the radius of the 'water cone', h and r are variables.

$$\text{Vol. of water} = V = \frac{1}{3} \pi r^2 h$$

$$\text{From similar triangles, } \frac{r}{h} = \frac{3}{5}$$

$$\therefore r = \frac{3h}{5}$$

$$\therefore V = \frac{1}{3} \pi \cdot \frac{9h^2}{25} h = \frac{3\pi h^3}{25}$$

$$\therefore \frac{dV}{dt} = \frac{3\pi}{25} \cdot 3h^2 \frac{dh}{dt}$$

$$= \frac{9\pi h^2}{25} \frac{dh}{dt}$$

$$\text{By hypothesis, } \frac{dV}{dt} = -3$$

$$\therefore \frac{dh}{dt} = \frac{-3 \times 25}{9\pi h^2}$$

$$= \frac{-25}{3\pi h^2}$$

$$\therefore \frac{dh}{dt} \text{ required} = \frac{-25}{3\pi \times 9} = \frac{-25}{27\pi} \text{ in per minute.}$$

Exercises

1 The distance s ft. traversed by a particle at time t is given by the formula $s = t^3 - 6t^2 - 4t - 8$. Find when the acceleration is 0 and the velocity at that instant.

Ans. $t = 2$; $v = 16$ per sec.

2 A stone is dropped into a pond. Waves in the form of circles are generated and the radius of the outermost ripple increases at the rate of 2 inches per sec. How fast is the area increasing 1) when the radius = 5 inches 2) after 5 secs.?

Ans. 1) 20π , 2) 40π

3 A body is projected vertically upwards and travels a distance s ft. above the point of projection in time t secs,

where $s = 160t - 16t^2$. When does the velocity vanish? What is the greatest height reached by the body?

Ans. 5 secs., 400 ft.

4 The distance s ft. travelled by a particle moving along a straight line in t secs. is given by the formula $s = 2t^3 - 9t^2 + 12t + 6$. Find when its acceleration is zero and the velocity at that instant. Ans. $1\frac{1}{2}$ secs., $-\frac{3}{2}$ ' / sec.

5 Express the following in the form of an equation involving the differential coefficient:—

(1) The area A of a circle increases k times as fast as the radius.

(2) The volume V of a soap bubble increases 3 times as fast as the radius.

(3) The rate of increase of y with respect to x is proportional to x^2 .

(4) The rate of increase of V with respect to h is inversely proportional to h .

(5) The rate of decrease of volume V of a sphere is directly proportional to the radius.

Ans. 1) $\frac{dA}{dt} = k \frac{dr}{dt}$ 2) $\frac{dV}{dt} = 3 \frac{dr}{dt}$ 3) $\frac{dy}{dx} = kx^2$

4) $\frac{dV}{dh} = \frac{k}{h}$ 5) $\frac{dV}{dt} = -kr$.

6 The volume of a spherical balloon steadily decreases at the constant rate of 3 cu. in. per minute. What is the rate of decrease of the radius when the radius is 5 in.?

Ans. $\frac{3''}{100\pi}$ / min.

7 A circular blot of ink in a blotting paper increases in area in such a way that the radius r in. at time t secs. is given by $r = 2t^2 - \frac{t^3}{4}$. What is the rate of increase of the

area when (1) $t = 4$. (2) $t = \frac{16}{3}$. Ans. (1) 128π ; (2) 0

8 A drop of ink spreads over a blotting paper so that the area of the blot which is circular increases at the rate of 2 sq. cm. per second. Find the rate at which the radius of the blot is increasing when the area is 5 sq. cm

Ans. $\frac{1}{\sqrt{5\pi}}$

9 The volume of a sphere is increasing at 4π cu.cm. per second. Find the rate of increase of the radius and of the surface area when the volume is 288π cu.cm.

[The surface area of a sphere is $4\pi r^2$.] Ans. $\frac{1}{36}$, $\frac{4\pi}{3}$

10 A circular patch of oil spreads on water, the area growing at the rate of 16 sq. inches per minute. How fast is the radius increasing when the radius is 12 inches?

Ans. $\frac{2''}{3\pi}$ / min.

11 A metal plate in the form of a circle expands by heat so that the radius increases uniformly at the rate of 1'' per minute. Find the rate of increase of area of the plate when the radius is 6''.

Ans. 12π sq.in./min.

12 An electric arc lamp B is hung directly over a certain point A on a straight road, $AB = 10$ ft. A man 5 ft. high is walking away from A along the road at the constant speed of 3 miles per hour. Calculate (1) the rate at which his shadow lengthens, (2) the rate at which the end of the shadow is moving. Ans. (1) 3 mph. (2) 6 mph.

13 A man 6 ft. tall moves away from a source of light 20 ft. above the ground, his rate of walking being 4 miles per hour. What is the rate at which the end of his shadow is moving?

Ans. $\frac{40}{3}$

14 A football bladder is being inflated with air at the rate of 36 cubic inches per second. Find the rate at which

the radius is increasing when the radius is 4 inches.

Ans. $\frac{9}{16\pi}$

15 If, with the usual notation, $s = 4t^3 - 6t^2 + t - 7$, what is the acceleration when $t = 2$ secs. and what is the speed at that instant?

Ans. 36; 25

16 The cross-section of a trough is an equilateral triangle. If the trough is 15 ft. long, and if water flows into it at the steady rate of 25 cub. ft. per sec., find the rate at which the level of the water is rising when the depth of the water is 5 ft.

Ans. $\frac{\sqrt{3}}{6}$

17 A ladder 20 ft. long rests with its ends on a smooth horizontal floor and against a smooth vertical wall. If the lower end is moved at the rate of 5 ft. per sec., find the rate at which the upper end moves, when the lower end is 12 ft. from the wall.

Ans. $\frac{16}{3}$

18 A point moves along a straight line so that its distance x from a fixed point O on it at any time t is given by the formula $x = 3 \sin t + 2 \cos t$. Find its acceleration after a time $= \frac{\pi}{3}$ secs. Show also that the acceleration is numerically always equal to the distance from O .

Ans. $-\left(\frac{3\sqrt{3}}{2} + 1\right)$

19 When brakes are applied to a moving car, the car travels a distance of s ft. in time t , given by $s = 20t - 4t^2$. When does the car stop?

Ans. $\frac{5}{2}$

20 If $s = at^3 + bt$ where a and b are constants, and s and t have their usual meanings, find a and b given that when $t = 3$, the velocity is zero and the acceleration is 14 units.

Ans. $\frac{7}{3}, -21$

21 If $s = kv^2$, where k is a constant, and v is the velocity of a particle attained in a distance s , show that the acceleration is constant.

22 A square plate expands uniformly, the side increasing at the rate of 0.2 in. per sec. What is the rate at which the area is increasing when the side is 10 in.?

Ans. 4

23 Water flows out of a conical filter at the steady rate of 4 cu. in. per second. The height of the filter is 3 in. and the radius of the circular base is 4 in. Find the rate at which the level of the water falls when the depth of the water is 2 inches.

Ans. $\frac{9}{16\pi}$

24 At constant temperature, the volume v and pressure p of a gas are related by the formula $pv = k$, k being a constant. Find the rate of increase of v with respect to p when 1) $v = 3$. 2) $p = 4$, given that $v = 4$ when $p = 6$.

Ans. $-\frac{3}{8}, -\frac{3}{8}$

25 Show that $\frac{dv}{dt} = v \frac{dv}{ds}$

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40

4

INTEGRATION

33 Integration We have considered in the previous chapters the method of finding the differential coefficients of some elementary functions. We shall now apply ourselves to the converse problem, i.e. given the differential coefficient of a function, to find the function. This process is called 'Integration' and the function obtained is called the **Integral** of the given function.

In general if the given function is $f(x)$ the problem of integration is to find another function, say $F(x)$ such that the differential coefficient of $F(x)$ is $f(x)$, i.e. $F'(x) = f(x)$.

We know that the differential coefficient of x^2 with respect to x is $2x$. This can be stated in the equivalent form "the integral of $2x$ with respect to x is x^2 ". Similarly the differential coefficient of $\sin x$ with respect to x is $\cos x$, i.e. the integral of $\cos x$ with respect to x is $\sin x$.

If one such function $F(x)$ is found then $F(x) + c$ (where c is any constant) will also be an integral, for $F(x)$ and $F(x) + c$ have the same differential coefficient, and hence the differential coefficient of $F(x) + c$ will also be $f(x)$. Thus the integral of $f(x)$ is not unique but the various answers will all differ by a constant.

34 Notation The symbol $\int$ is used to denote the integral of a function. e.g. $\int f(x)dx$ means and is read as "the integral of $f(x)$ with respect to x ". The symbols $\frac{d}{dx}$ and $\int$ are connected thus: If $\frac{dy}{dx} = f(x)$ then $y = \int f(x)dx + c$. When $f(x)$ is an elementary function the integral can be obtained by inspection.

35 To find $\int x^n dx$

We know that if we differentiate any power of x we get another power of x which is one less than the original. Hence if we should get x^n as the result of differentiation we should take x^{n+1} and differentiate. x^{n+1} gives, on differentiation, $(n+1)x^n$. If, therefore we take $\frac{x^{n+1}}{n+1}$ and then differentiate we get x^n . Therefore $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, where c is any constant.

This result is true for all values of n except $n = -1$. If $n = -1$, the R.H.S. becomes meaningless.

36 To find $\int \sin x dx$

We know that $\frac{d}{dx}(\cos x) = -\sin x$. Hence to get $\sin x$ as the result of differentiation we should take $-\cos x$ and differentiate.

Hence $\int \sin x dx = -\cos x + c$.

37 To find $\int \cos x dx$

We know that $\frac{d}{dx}(\sin x) = \cos x$. Hence $\int \cos x dx = \sin x + c$.

38 To find $\int (ax + b)^n dx$

We know that $\frac{d}{dx}(ax + b)^{n+1} = (n+1)a(ax + b)^n$

$$\therefore \int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{(n+1)a} + c.$$

39 To find $\int \sin mx dx$

We know that $\frac{d}{dx}(\cos mx) = -m \sin mx$

$$\therefore \int \sin mx dx = -\frac{\cos mx}{m} + c.$$

40 To find $\int \cos nx \, dx$

We know that $\frac{d}{dx} (\sin nx) = n \cos nx$

$$\therefore \int \cos nx \, dx = \frac{\sin nx}{n} + c.$$

Worked Examples

1 $\int k \, dx = kx + c$, k being a constant 2 $\int x \, dx = \frac{x^2}{2} + c$

3 $\int (x+3)^2 \, dx = \frac{(x+3)^3}{3} + c$

4 $\int (x-7)^{10} \, dx = \frac{(x-7)^{11}}{11} + c$

5 $\int \sqrt{x} \, dx = \frac{2}{3} x^{\frac{3}{2}} + c$ 6 $\int x^{\frac{2}{3}} \, dx = \frac{3}{5} x^{\frac{5}{3}} + c$

7 $\int \frac{1}{x^4} \, dx = \int x^{-4} \, dx = \frac{x^{-3}}{-3} + c = -\frac{1}{3x^3} + c$

8 $\int \frac{1}{\sqrt{x}} \, dx = \int x^{-\frac{1}{2}} \, dx = 2\sqrt{x} + c$

9 $\int (2x-1)^{10} \, dx = \frac{(2x-1)^{11}}{11(2)} = \frac{(2x-1)^{11}}{22} + c$

10 $\int (1-3x)^5 \, dx = \frac{(1-3x)^6}{-18} + c$

11 $\int \sin 3x \, dx = -\frac{\cos 3x}{3} + c$

12 $\int \cos 5x \, dx = \frac{\sin 5x}{5} + c$

13 $\int \sin (3x+4) \, dx = -\frac{\cos (3x+4)}{3} + c$

14 $\int \cos (2-7x) \, dx = \frac{\sin (2-7x)}{-7} + c$

Exercises

Find the integrals with respect to x of the following:

- 1 x^{100} 2 x^{75} 3 $(x-1)^{200}$ 4 $(x+3)^{\frac{25}{2}}$ 5 $(x-7)^{\frac{1}{2}}$ 6 $(x-3)^{\frac{1}{2}}$ 7 $\frac{1}{(x-100)^2}$ 8 $\frac{1}{(x-1)^3}$ 9 $\frac{1}{\sqrt{x+3}}$ 10 $\frac{1}{(x+1)\sqrt{x+1}}$ 11 $(3x-1)^{18}$ 12 $(1-4x)^{12}$ 13 $(3+2x)^{\frac{4}{3}}$ 14 $\frac{1}{\sqrt{1+5x}}$ 15 $\frac{3}{(2-3x)^{\frac{1}{2}}}$ 16 $\frac{5}{(7-4x)^3}$ 17 $\sin \frac{x}{3}$ 18 $\cos \frac{x}{2}$ 19 $\sin (5x-1)$ 20 $\cos (3x+1)$ 21 $\sin (2-3x)$ 22 $\cos (5-4x)$ 23 $\sin \left(\frac{x}{3} + 2\right)$ 24 $\cos \left(\frac{x}{4} - 7\right)$

- Ans. 1 $\frac{x^{101}}{101} + c$ 2 $\frac{x^{76}}{76} + c$ 3 $\frac{(x-1)^{201}}{201} + c$ 4 $\frac{2(x+3)^{\frac{27}{2}}}{27} + c$ 5 $\frac{3}{2}(x-7)^{\frac{3}{2}} + c$ 6 $\frac{3}{5}(x-3)^{\frac{5}{2}} + c$ 7 $-\frac{1}{x-100} + c$ 8 $-\frac{1}{2(x-1)^2} + c$ 9 $2\sqrt{x+3} + c$ 10 $-\frac{2}{\sqrt{x+1}} + c$ 11 $\frac{(3x-1)^{19}}{57} + c$ 12 $\frac{(1-4x)^{13}}{-52} + c$ 13 $\frac{5(3+2x)^{\frac{8}{3}}}{18} + c$ 14 $\frac{2\sqrt{1+5x}}{5} + c$ 15 $-\frac{3}{2}(2-3x)^{\frac{3}{2}} + c$ 16 $\frac{5}{8(7-4x)^2} + c$ 17 $-3 \cos \frac{x}{3} + c$ 18 $2 \sin \frac{x}{2} + c$ 19 $-\frac{\cos (5x-1)}{5} + c$ 20 $\frac{\sin (3x+1)}{3} + c$ 21 $\frac{\cos (2-3x)}{3} + c$ 22 $-\frac{\sin (5-4x)}{4} + c$ 23 $-3 \cos \left(\frac{x}{3} + 2\right) + c$ 24 $4 \sin \left(\frac{x}{4} - 7\right) + c$

41 From the foregoing examples it is now clear that for elementary functions the integral is obtained by using the results already arrived at in the chapters on the Differential Calculus. This list of integrals can be extended thus:—

- 1) If $y = \tan x$, we know that $\frac{dy}{dx} = \sec^2 x$
 $\therefore$ we can say that $\int \sec^2 x dx = \tan x + c$.
- 2) If $y = \cot x$ then we know that $\frac{dy}{dx} = -\operatorname{cosec}^2 x$
 $\therefore$ we can say that $\int \operatorname{cosec}^2 x dx = -\cot x + c$.
- 3) If $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$
 $\therefore$ we can say that $\int \sec x \tan x dx = \sec x + c$.
- 4) If $y = \operatorname{cosec} x$ then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$
 $\therefore$ we can say that $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$.

Exercises

Find the integrals with respect to x of;

- 1) $\sec^2 3x$ 2) $\operatorname{cosec}^2 4x$ 3) $\sec 4x \tan 4x$ 4) $\frac{\sin 3x}{\cos^2 3x}$
 5) $\frac{\cos 2x}{\sin^2 2x}$ 6) $\frac{1}{\cos^2 5x}$ 7) $\frac{1}{\sin^2 3x}$ 8) $\operatorname{cosec} 3x \cot 3x$

- Ans. 1) $\frac{\tan 3x}{3} + c$ 2) $-\frac{\cot 4x}{4} + c$ 3) $\frac{\sec 4x}{4} + c$
 4) $\frac{\sec 3x}{3} + c$ 5) $-\frac{\operatorname{cosec} 2x}{2} + c$ 6) $\frac{\tan 5x}{5} + c$
 7) $-\frac{\cot 3x}{3} + c$ 8) $-\frac{\operatorname{cosec} 3x}{3} + c$

42 To show that $\int (u + v - w) dx = \int u dx + \int v dx - \int w dx + c$ where u , v and w are functions of x .

Differentiate the expression $\int u dx + \int v dx - \int w dx + c$ with respect to x . We get, $\frac{d}{dx} [\int u dx + \int v dx - \int w dx + c]$

$$= \frac{d}{dx} (\int u dx) + \frac{d}{dx} (\int v dx) - \frac{d}{dx} (\int w dx) + \frac{dc}{dx},$$

from Art. 12

$$= u + v - w + 0, \text{ from definition,}$$

$$= u + v - w.$$

$$\int (u + v - w) dx = \int u dx + \int v dx - \int w dx + c,$$

from definition,

i.e. the integral of the algebraic sum of any finite number of functions is equal to the algebraic sum of the integrals of the functions.

43 Change of the Independent variable or the method of substitution

$$\text{To show that } \int f(x) dx = \int f(x) \frac{dx}{dt} dt.$$

$$\text{Let } y = \int f(x) dx.$$

$$\text{Then } \frac{dy}{dx} = f(x) \text{ by definition.}$$

$$\text{We know that } \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \text{ (Chain Rule)}$$

$$= f(x) \frac{dx}{dt}.$$

$\therefore$ integrating both sides with respect to t , we get

$$y = \int f(x) \frac{dx}{dt} dt$$

$$\text{i.e. } \int f(x) dx = \int f(x) \frac{dx}{dt} dt.$$

Note: From this result it is clear that we can replace dx by $\frac{dx}{dt} dt$ where t is a variable connected with x in some way and then proceed to find $\int f(x) \frac{dx}{dt} dt$. This artifice is often useful and reduces the problem to a simpler one.

44 To find $\int [f(x)]^n f'(x) dx$

$$\text{Let } f(x) = t.$$

$$\text{Then } f'(x) \frac{dx}{dt} = 1$$



$$\begin{aligned}\int [f(x)]^n f'(x) dx &= \int [f(x)]^n f'(x) \frac{dx}{dt} dt, \text{ by Art. 43} \\ &= \int t^n (1) dt \\ &= \frac{t^{n+1}}{n+1} + c \\ &= \frac{[f(x)]^{n+1}}{n+1} + c.\end{aligned}$$

Note: $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c.$

Worked Examples

1 Find $\int (2x+1)^7 dx.$

Put $2x+1 = t$

Then $2 \frac{dx}{dt} = 1$

$$\begin{aligned}\therefore \int (2x+1)^7 dx &= \int (2x+1)^7 \frac{dx}{dt} dt \\ &= \int t^7 \left(\frac{1}{2}\right) dt = \frac{1}{2} \frac{t^8}{8} + c \\ &= \frac{(2x+1)^8}{16} + c.\end{aligned}$$

Note: After some practice the integrals of this type can be written down by inspection,

2 Find $\int (3x^2+x-1)^6 (6x+1) dx.$

Put $3x^2+x-1 = t$

Then $(6x+1) \frac{dx}{dt} = 1$

$$\begin{aligned}\therefore \int (3x^2+x-1)^6 (6x+1) dx &= \int (3x^2+x-1)^6 (6x+1) \frac{dx}{dt} dt \\ &= \int t^6 (1) dt \\ &= \frac{t^7}{7} = \frac{(3x^2+x-1)^7}{7} + c.\end{aligned}$$

3 Find $\int \sin^3 x \cos x dx.$

Put $\sin x = t$

Then $\cos x \frac{dx}{dt} = 1$

$$\begin{aligned}\therefore \int \sin^3 x \cos x dx &= \int \sin^2 x \cos x \frac{dx}{dt} dt \\ &= \int t^2 \cdot (1) dt = \frac{t^3}{3} + c \\ &= \frac{\sin^3 x}{3} + c.\end{aligned}$$

4 Find $\int \tan^4 \theta \sec^2 \theta d\theta.$

Put $\tan \theta = t$

Then $\sec^2 \theta \frac{d\theta}{dt} = 1$

$$\begin{aligned}\therefore \int \tan^4 \theta \sec^2 \theta d\theta &= \int \tan^4 \theta \sec^2 \theta \frac{d\theta}{dt} dt \\ &= \int t^4 (1) dt = \frac{t^5}{5} + c \\ &= \frac{\tan^5 \theta}{5} + c.\end{aligned}$$

5 Find $\int \sin^2 x dx.$

We know that $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$\begin{aligned}\therefore \int \sin^2 x dx &= \int \frac{1 - \cos 2x}{2} dx = \frac{1}{2} \int (1 - \cos 2x) dx \\ &= \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) + c.\end{aligned}$$

6 Find $\int \sin 5x \cos 3x dx$

We know that $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$

$$\begin{aligned}\therefore \int \sin 5x \cos 3x dx &= \int \frac{\sin 8x + \sin 2x}{2} dx \\ &= \frac{1}{2} \left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2} \right] + c.\end{aligned}$$

7 Find $\int \frac{xdx}{\sqrt{a^2-x^2}}$

Let $x = a \sin \theta$

Then $\frac{dx}{d\theta} = a \cos \theta$

$$\begin{aligned}
 \therefore \int \frac{x dx}{\sqrt{a^2 - x^2}} &= \int \frac{x}{\sqrt{a^2 - x^2}} \frac{dx}{d\theta} \cdot d\theta \\
 &= \int \frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \cdot a \cos \theta d\theta \\
 &= a^2 \int \frac{\sin \theta \cos \theta}{a \cos \theta} d\theta \\
 &= a \int \sin \theta d\theta = a(-\cos \theta) + c \\
 &= -a \sqrt{1 - \frac{x^2}{a^2}} + c \\
 &= -\sqrt{a^2 - x^2} + c.
 \end{aligned}$$

Exercises

Find the integral with respect to x of:

- 1 $\sin^2 x \cos x$ 2 $\cos^3 x \sin x$ 3 $\sin 3x \cos 4x$
 4 $\sin 4x \cos 3x$ 5 $\sin 2x \sin 4x$ 6 $\cos 3x \cos 5x$
 7 $\frac{\sin^3 x}{\cos^5 x}$ 8 $\tan^2 x$ 9 $\tan^4 x$
 10 $\sin^3 x$ 11 $\cos^3 x$ 12 $x\sqrt{x^2-1}$
 13 $\frac{x}{\sqrt{16+x^2}}$ 14 $\frac{x+2}{\sqrt{x^2+4x+3}}$ 15 $\frac{1}{\sqrt{a^2-x^2}}$
 16 $\frac{1}{\sqrt{2x-x^2}}$ 17 $\frac{x^2}{\sqrt{64-x^6}}$ 18 $x\sqrt{16-x^2}$
 19 $\cot^2 x$ 20 $\sec^2 2x$ 21 $\operatorname{cosec}^2 3x$
 22 $\sec^2 \frac{x}{2}$ 23 $\operatorname{cosec}^2 \frac{x}{3}$ 24 $(x^2+x-1)^{10}(2x+1)$
 25 $(x^2+2x+3)^{100}(x+1)$ 26 $(1+3x+x^3)^4(x^2+1)$
 27 $(1-x^2)^5 x$ 28 $x\sqrt{a^2+x^2}$ 29 $\frac{4x^3}{\sqrt{16-x^4}}$
 30 $\frac{\sin x}{(a-b \cos x)^4}$ 31 $\frac{\cos x}{\sqrt[3]{1-9 \sin x}}$ 32 $\frac{x}{(1-x^2)^3}$
 33 $4x^3 \cos(x^4)$ 34 $x^2 \sin(x^3)$ 35 $\frac{\cos 2x}{\sqrt{\sin 2x}}$

$$\begin{aligned}
 36 \sec^3 x \tan x & \quad 37 \tan^3 x \sec^2 x & 38 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \\
 39 \frac{(\tan^{-1} x)^2}{1+x^2} & \quad 40 \frac{\cos \sqrt{x}}{\sqrt{x}}
 \end{aligned}$$

$$\text{Ans. } 1 \frac{\sin^3 x}{3} + c$$

$$2 \frac{\cos^4 x}{4} + c$$

$$3 \frac{1}{2} \left[-\frac{\cos 7x}{7} + \cos x \right] + c \quad 4 \frac{1}{2} \left[-\frac{\cos 7x}{7} - \cos x \right] + c$$

$$5 \frac{1}{2} \left[\frac{\sin 2x}{2} - \frac{\sin 6x}{6} \right] + c \quad 6 \frac{1}{2} \left[\frac{\sin 8x}{8} + \frac{\sin 2x}{2} \right] + c$$

$$7 \frac{\tan^4 x}{4} + c$$

$$8 \tan x - x + c$$

$$9 \frac{\tan^3 x}{3} - \tan x + x + c$$

$$10 \frac{1}{4} \left[-3 \cos x + \frac{\cos 3x}{3} \right] + c$$

$$11 \frac{1}{4} \left[\frac{\sin 3x}{3} + 3 \sin x \right] + c$$

$$12 \frac{(x^2-1)^{\frac{3}{2}}}{3} + c$$

$$13 \sqrt{16+x^2} + c$$

$$14 \sqrt{x^2+4x+3} + c$$

$$15 \sin^{-1} \frac{x}{a} + c$$

$$16 \sin^{-1}(x-1) + c$$

$$17 \frac{1}{8} \sin^{-1} \frac{x^3}{8} + c$$

$$18 -\frac{1}{8} (16-x^2)^{\frac{3}{2}} + c$$

$$19 -\cot x - x + c$$

$$20 \frac{\tan 2x}{2} + c$$

$$21 -\frac{\cot 3x}{3} + c$$

$$22 2 \tan \frac{x}{2} + c$$

$$23 -3 \cot \frac{x}{3} + c$$

$$24 \frac{(x^2+x-1)^{11}}{11} + c$$

$$25 \frac{(x^2+2x+3)^{101}}{202} + c$$

$$26 \frac{(1+3x+x^3)^5}{15} + c$$

$$27 \frac{-(1-x^2)^6}{12} + c$$

$$28 \frac{(a^2+x^2)^{\frac{3}{2}}}{3} + c$$

$$29 -2(16-x^4)^{\frac{1}{4}} + c$$

$$30 \frac{1}{3b(a-b \cos x)^3} + c$$

$$31 \quad -\frac{(1-9\sin x)^{\frac{2}{3}}}{6} + c$$

$$33 \quad \sin(x^4) + c \quad \text{stop}$$

$$35 \quad \sqrt{\sin 2x} + c$$

$$37 \quad \frac{\tan^4 x}{4} + c$$

$$39 \quad \frac{(\tan^{-1}x)^3}{3} + c$$

$$32 \quad \frac{1}{4(1-x^2)^2} + c$$

$$34 \quad \frac{-\cos(x^3)}{3} + c$$

$$36 \quad \frac{\sec^3 x}{3} + c$$

$$38 \quad \frac{-(\cos^{-1}x)^2}{2} + c$$

$$40 \quad 2 \sin \sqrt{x} + c$$

45 Integration by parts

If u and v are functions of x , we know from earlier chapters that $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$.

Integrating both sides with respect to x , we get,

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx,$$

$$\therefore \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

$$\text{Let } u = f(x) \text{ and } \frac{dv}{dx} = \phi(x).$$

Then the formula above can be written as

$$\int f(x) \phi(x) dx = f(x) \int \phi(x) dx - \int [f'(x) \int \phi(x) dx] dx.$$

In words this formula can be stated this way:

The integral of a product of two functions = the first function $\times$ the integral of the second function minus the integral of the product of the differential coefficient of the first and the integral of the second.

Note: The formula can be remembered also as $\int u dv = uv - \int v du$.

Worked Examples

- 1 Find $\int x \cos x dx$.

Take x as the first function and $\cos x$ as the second function.

Applying the formula above, we get

$$\begin{aligned} \int x \cos x dx &= x(\sin x) - \int 1(\sin x) dx \\ &= x \sin x + \cos x + c. \end{aligned}$$

- 2 Find $\int x^2 \sin x dx$.

Take x^2 as the first function and $\sin x$ as the second function.

$$\begin{aligned} \text{We get, } \int x^2 \sin x dx &= x^2(-\cos x) - \int 2x(-\cos x) dx \\ &= -x^2 \cos x + 2 \int x \cos x dx \\ &= -x^2 \cos x + 2(x \sin x + \cos x) \\ &+ c, \text{ from the last example.} \end{aligned}$$

- 3 Find $\int \sin^{-1} x dx$.

Take $\sin^{-1} x$ as the first function and 1 as the second function.

$$\begin{aligned} \int \sin^{-1} x dx &= \int \sin^{-1} x (1) dx \\ &= \sin^{-1} x (x) - \int \frac{1}{\sqrt{1-x^2}} x dx. \end{aligned}$$

By putting $x = \sin \theta$, we can show that

$$\int \frac{x dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2}.$$

(See worked example 7, Art. 44)

$$\therefore \int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + c.$$

Exercises

Integrate with respect to x :

- 1 $x^2 \cos x$
- 2 $x^3 \cos x$
- 3 $x \cos^2 x$
- 4 $x \sin 2x$
- 5 $x \cos 3x$
- 6 $x \sin x \sin 3x$
- 7 $x \cos 3x \cos x$
- 8 $x(x-2) \cos x$.

- Ans. 1 $x^2 \sin x + 2x \cos x - 2 \sin x + c$
 2 $x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$
 3 $\frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + c$
 4 $\frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} + c$
 5 $\frac{x \sin 3x}{3} + \frac{\cos 3x}{9} + c$
 6 $\frac{1}{4} \left(x \sin 2x + \frac{\cos 2x}{2} \right) - \frac{1}{8} \left(x \sin 4x + \frac{\cos 4x}{4} \right) + c$
 7 $\frac{1}{4} \left(\frac{x \sin 4x}{2} + \frac{\cos 4x}{8} + x \sin 2x + \frac{\cos 2x}{2} \right) + c$
 8 $(x^2 - 2x - 2) \sin x + 2(x - 1) \cos x + c.$

$$0 + \infty$$

$$0 - \infty$$

$$0 - 4$$

$$0 + 4$$

$$0 + \infty$$

$$0 - \infty$$

$$0 - 4$$

$$0 + 4$$

CHAPTER VII

DEFINITE INTEGRALS

46 **Definition** If $F(x)$ is an integral of $f(x)$ which exists for all values of x in the interval a to b , we define $\int_a^b f(x) dx$ as $F(b) - F(a)$. This is called the **Definite Integral** of $f(x)$ over the interval a to b . 'a' is called the lower limit and 'b' is called the upper limit. It is read as integral from 'a' to 'b' of $f(x)dx$. Its value is a number independent of x . It is called definite because any constant c taken along with $F(x)$ as the constant of integration disappears on substituting the upper and lower limits i.e. if $F(x) + c$ is the indefinite integral, we get on substitution $F(b) + c - [F(a) + c] = F(b) - F(a)$.

Worked Examples

$$1 \int_2^3 x^2 dx = \left(\frac{x^3}{3} \right)_2^3 = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}.$$

$$2 \int_1^2 \sqrt{x} dx = \left(\frac{2}{3} x^{\frac{3}{2}} \right)_1^2 = \frac{2}{3} \left(2^{\frac{3}{2}} - 1 \right) = \frac{2}{3} (2\sqrt{2} - 1).$$

$$3 \int_0^{\frac{\pi}{2}} \sin x dx = \left(-\cos x \right)_0^{\frac{\pi}{2}} \\ = \left(-\cos \frac{\pi}{2} \right) - (-\cos 0) = 0 + 1 = 1.$$

$$4 \int_2^3 dx = \left(x \right)_2^3 = 3 - 2 = 1.$$

Exercises

Evaluate the following definite integrals:

$$1) \int_0^2 x^3 dx \quad 2) \int_1^2 x^4 dx \quad 3) \int_0^3 (x^2 + 3) dx$$

$$4) \int_0^1 (x+1)^2 dx \quad 5) \int_{-1}^1 x^2 dx \quad 6) \int_{-2}^0 x^4 dx$$

$$7) \int_0^{\frac{\pi}{2}} \cos x dx \quad 8) \int_0^{\frac{\pi}{2}} \sin^2 x dx \quad 9) \int_0^{\frac{\pi}{2}} \cos^2 x dx$$

$$10) \int_0^{\frac{\pi}{4}} \sec^2 x dx \quad 11) \int_a^b \frac{dx}{\sqrt{x}} \quad 12) \int_0^4 (2x+1)^{\frac{3}{2}} dx$$

$$13) \int_0^{\pi} \sin 3x \cos x dx \quad 14) \int_0^{\pi} \cos 5x \cos 3x dx$$

$$15) \int_0^{\frac{\pi}{2}} \sin 2x \cos x dx \quad 16) \int_{-1}^1 (x^3 - x^2 - x + 3) dx.$$

$$17) \text{ Prove that } \int_a^b f(x) dx = - \int_b^a f(x) dx.$$

$$18) \text{ Prove that } \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$19) \text{ Prove that } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Ans: 1) 4 2) $\frac{31}{5}$ 3) 18 4) $\frac{7}{3}$ 5) $\frac{2}{3}$

6) $\frac{32}{5}$ 7) 1 8) $\frac{\pi}{4}$ 9) $\frac{\pi}{4}$ 10) 1 11) $2(\sqrt{b} - \sqrt{a})$

12) $\frac{242}{5}$ 13) 0 14) 0 15) $\frac{2}{3}$ 16) $\frac{16}{3}$

47 Fundamental Theorem

In the previous Article we defined the Definite Integral $\int_a^b f(x) dx$ as equal to $F(b) - F(a)$ where $F(x)$ is a function whose differential coefficient with respect to x is $f(x)$.

That is, if $\frac{d}{dx} F(x) = f(x)$ then $\int_a^b f(x) dx = F(b) - F(a)$.

Thus the value of the Definite Integral is got by integrating the given function $f(x)$, substituting the upper limit b and the lower limit a for x in the integral and then subtracting the latter from the former. Thus the value depends on the function $f(x)$ and the constants a and b .

But there is another aspect of the Definite Integral which is of great importance in Mathematics. It can be shown that the Definite Integral is the limit of the sum of an infinite number of terms each of which tends to zero. A good understanding of this aspect will enable the student to make intelligent use of it in all branches of Mathematics like Statics, Dynamics etc. It is very often stated briefly as "Integration is equivalent to Summation" and is contained in the following Fundamental Theorem:

Let the interval $[a, b]$ be divided into n equal parts each of length h , so that $nh = b - a$. Then the Fundamental theorem of the Integral Calculus states that

$$\lim_{n \rightarrow \infty} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)] = \int_a^b f(x) dx.$$

48 Application to Areas

Let a function $f(x)$ be positive and continuous in the interval a to b . If the graph of $y = f(x)$ is drawn then it lies above the x -axis. Then by application of the Fundamental Theorem of Article 47 it can be shown that the area included between the curve $y = f(x)$, the x -axis and

the ordinates $x = a$ and $x = b$ is given by the Definite Integral $\int_a^b f(x) dx$.

Let PQ be the curve $y = f(x)$. Let $OA = a$ and $OB = b$.

Let the segment AB be divided into n equal parts each of length h so that $nh = b - a$. At the points of division draw ordinates to meet the curve at various points. Through

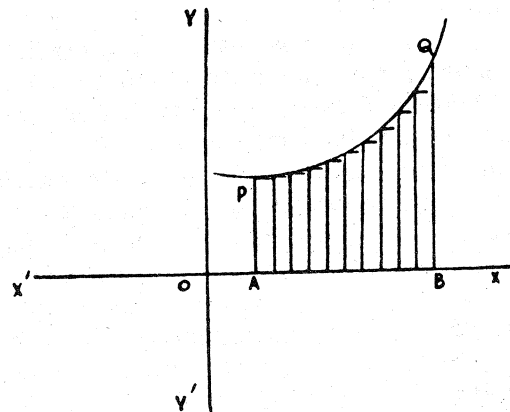


Fig. 14

these points draw parallels to the x -axis as in the figure. We have then a number of rectangles. The area of the first rectangle is $hf(a)$, that of the second is $hf(a+h)$, that of the third is $hf(a+2h)$, and so on.

$\therefore$ the sum of the areas of the rectangles is $h[f(a) + f(a+h) + f(a+2h) + \dots \text{to } n \text{ terms}]$.

When $n \rightarrow \infty$ i.e. $h \rightarrow 0$ the points of division on the x -axis come closer and closer to one another and the sum of the areas of the rectangles $\rightarrow$ the area $PABQ$.

$$\begin{aligned} \therefore \text{area } PABQ &= \lim_{n \rightarrow \infty} [\text{sum of the areas of the rectangles}] \\ &= \lim_{n \rightarrow \infty} h[f(a) + f(a+h) + \dots \text{to } n \text{ terms}] \\ &= \int_a^b f(x) dx, \text{ by the Fundamental Theorem.} \end{aligned}$$

Worked Examples

- 1) Find the area bounded by the x -axis, and the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

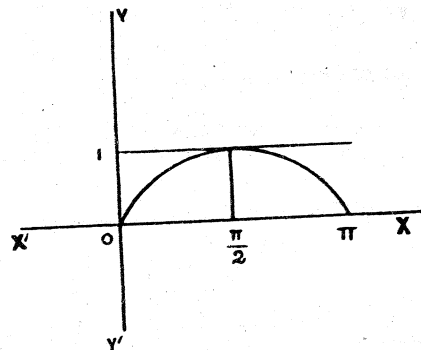


Fig. 15

$$\begin{aligned} \text{Area required} &= \int_0^{\pi} \sin x dx = (-\cos x)_0^{\pi} \\ &= -\cos \pi - (-\cos 0) = 1 + 1 = 2 \text{ sq. units.} \end{aligned}$$

- 2) Prove that the area of a circle of radius a is πa^2 .

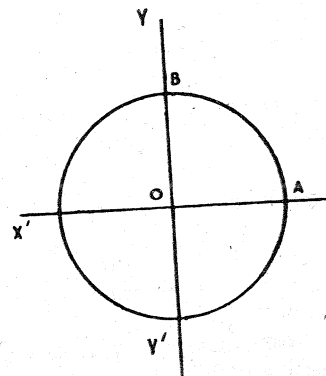


Fig. 16

Take the centre of the circle as the origin and any two perpendicular diameters as axes. Then the equation of the circle is

$$x^2 + y^2 = a^2,$$

$$\therefore y^2 = a^2 - x^2$$

$$\text{i.e. } y = \pm \sqrt{a^2 - x^2}.$$

Area of the circle = 4 area OAB ,

$$= 4 \int_0^a \sqrt{a^2 - x^2} dx.$$

To integrate $\sqrt{a^2 - x^2}$, put $x = a \sin \theta$

$$\frac{dx}{d\theta} = a \cos \theta.$$

$$\therefore \int \sqrt{a^2 - x^2} dx = \int \sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta d\theta$$

$$= a^2 \int \cos^2 \theta d\theta$$

$$= a^2 \int \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{a^2}{2} \left(\theta + \frac{\sin 2\theta}{2} \right)$$

$$= \frac{a^2}{2} (\theta + \sin \theta \cos \theta)$$

$$= \frac{a^2}{2} \left(\sin^{-1} \frac{x}{a} + \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} \right)$$

$$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2}.$$

$$\therefore \text{area required} = 4 \left[\frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} \right]_0^a$$

$$= 4 \left[\frac{a^2}{2} \cdot \frac{\pi}{2} \right] = \pi a^2.$$

- 3) Find the area between the curve $y = 7x^2$, the x -axis, $x = 1$ and $x = 3$.

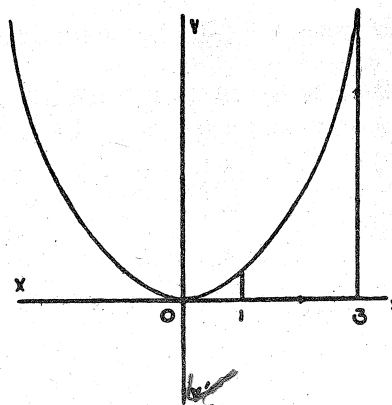


Fig. 17

$$\text{Area required} = \int_1^3 7x^2 dx$$

$$= \left(\frac{7x^3}{3} \right)_1^3 = 63 - \frac{7}{3} = \frac{182}{3} \text{ sq. units.}$$

- 4) Find the area between the curves $y = 2x^2 + 1$ and $y = x^2 + 5$.

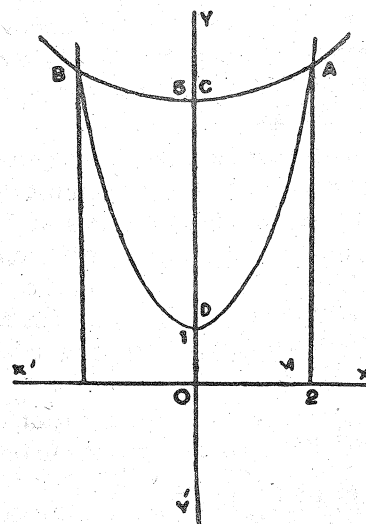


Fig. 18

Solving between the two given equations,

we get, $2x^2 + 1 = x^2 + 5$

i.e. $x^2 = 4$ i.e. $x = \pm 2$, and $y = 9$.

$\therefore$ the points of intersection of the curves are (2, 9) and (-2, 9).

Both the curves are symmetrical about the y -axis.

$\therefore$ area required = 2 area DAC

$$= 2 [\text{area } OMAC - \text{area } OMAD]$$

$$= 2 \left[\int_0^2 (x^2 + 5) dx - \int_0^2 (2x^2 + 1) dx \right]$$

$$= 2 \int_0^2 (4 - x^2) dx = 2 \left(4x - \frac{x^3}{3} \right)_0^2$$

$$= 2 \left(8 - \frac{8}{3} \right) = \frac{32}{3} \text{ sq. units.}$$

Note: 1. When the limits of integration are chosen, the lower limit a is the smaller value and the upper limit b is the bigger value.

2. If the curve lies below the x -axis between the limits $x = a$ and $x = b$ ($a < b$) then the area included between the given curve $y = f(x)$, $x = a$, $x = b$ and the x -axis will be found to be negative. In such cases, only the numerical value should be taken.

3. If the curve crosses the x -axis between the limits a and b , the areas for the two parts must be found separately, i.e. if the curve crosses the x -axis at the point $x = c$, then the area from a to c and the area from c to b must be found out separately. One of these will be found to be positive and the other negative. The required area is the sum of the two areas both taken with the positive sign.

4. If the curve is given by the equation $x = f(y)$ then the area included between the given curve, y -axis

$y = a$ and $y = b$ is given by $\int_a^b f(y) dy$.

Exercises

Find the area enclosed between the following curves, x -axis and the ordinates at the points given:

① $y = 3x$; $x = 1$, $x = 4$

② $y = 4x + 3$; $x = 2$, $x = 5$

③ $4y = x^2$; $x = 1$, $x = 3$

④ $y^2 = 8x$; $x = 4$, $x = 9$

⑤ $y = 2 - x$; $x = 1$, $x = \frac{3}{2}$

⑥ $y = x(x - 4)$; $x = 1$, $x = 2$.

Explain the negative answer.

⑦ $y = x(x + 4)$; $x = -2$, $x = -1$

⑧ $3y = 2 \sin x$; $x = 0$, $x = \frac{\pi}{2}$

⑨ $y = \sin 2x$; $x = 0$, $x = \frac{\pi}{4}$

⑩ $y = \sin x + \cos x$; $x = 0$, $x = \frac{\pi}{2}$

⑪ $y = 2 \sec^2 x$; $x = 0$, $x = \frac{\pi}{4}$

⑫ $y = 3 \sin x + 4 \cos x$; $x = 0$, $x = \frac{\pi}{2}$

⑬ $y = 9 - x^2$; $x = 1$, $x = 3$

⑭ $y = x^2 + 4x + 5$; $x = -2$, $x = 1$

⑮ $y = \sqrt{2x}$; $x = 0$, $x = 4$ ⑯ $y = x + \frac{5}{x^2}$, $x = 1$, $x = 3$

⑰ $y = 7 - \frac{4}{x^2}$; $x = 1$, $x = 2$

⑱ $y = 9 \sin \frac{x}{3}$; $x = 0$, $x = 3\pi$

19 Find the area bounded by the curve $\sqrt{x} + \sqrt{y} = 2$ and the coordinate axes

20 Find the area between the curves $y^2 = 6x$ and $x^2 = 6y$

21 Find the area between $y = x^2$ and the straight line $3y = 4x$

22 Find the area between the curves $y = 8 - x^2$ and $y = x^2$

23 Find the area between the curves $y = x^2$ and $y = x^3$

- 24 Find the area included between the curve $y = x^3$ and the line $y = 8x$ in the first quadrant
- 25 Find the area enclosed by the graph of $y = \cos x$ and the x -axis between the ordinates $x = 0$ and $x = \pi$
- 26 Find the area enclosed by the graph of $y = \sin x$ x -axis and the ordinates $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$
- 27 Find the area enclosed by the curve $y = x^2 - 1$, x -axis and the ordinates $x = 0$ and $x = 2$
- 28 Find the area enclosed by the curve $x = \cos y$ between the limits 1) $y = 0$ and $y = \frac{\pi}{2}$, 2) $y = -\pi$ and $y = \pi$
- 29 Find the area between one arch of the curve $y = 2 \sin x$ and the x -axis
- 30 Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$

Ans. ① $\frac{45}{2}$ ② 51 ③ $\frac{13}{6}$ 4 $\frac{76\sqrt{2}}{3}$ 5 $\frac{3}{8}$

6 $\frac{11}{3}$ 7 $\frac{11}{3}$ 8 $\frac{2}{3}$ 9 $\frac{1}{2}$ 10 2 11 2 12 7

13 $\frac{28}{3}$ 14 12 15 $\frac{16\sqrt{2}}{3}$ 16 $\frac{22}{3}$ 17 5 18 54

19 $\frac{8}{3}$ 20 12 21 $\frac{32}{81}$ 22 $\frac{64}{3}$ 23 $\frac{1}{12}$ 24 16

25 2 26 2 27 2 28 1; 4 29 4 30 9

49 Definite Integrals, Method of Substitution

When the method of Substitution is adopted in evaluating a Definite Integral, it is usual to change the limits also. The limits for the new variable are those values which correspond to the values a and b of the original variable.

Worked Examples

1 Find $\int_0^{\frac{\pi}{2}} \frac{\cos x dx}{1 + \sin^2 x}$

Put $\sin x = t$, then $\cos x \frac{dx}{dt} = 1$

When $x = 0$, $t = \sin 0 = 0$ and when $x = \frac{\pi}{2}$,

$t = \sin \frac{\pi}{2} = 1$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} \frac{\cos x dx}{1 + \sin^2 x} &= \int_0^1 \frac{\cos x}{1 + \sin^2 x} \frac{dx}{dt} dt \\ &= \int_0^1 \frac{1}{1 + t^2} dt \\ &= (\tan^{-1} t)_0^1 \\ &= (\tan^{-1} 1) - (\tan^{-1} 0) \\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4} \end{aligned}$$

2 Find $\int_0^{\frac{\pi}{2}} (1 + 2 \cos \theta)^3 \sin \theta d\theta$

Put $\cos \theta = t$, then $-\sin \theta \frac{d\theta}{dt} = 1$

When $\theta = 0$, $t = \cos 0 = 1$, and when $\theta = \frac{\pi}{2}$,

$t = \cos \frac{\pi}{2} = 0$.

$$\begin{aligned}
 &\therefore \int_0^{\frac{\pi}{2}} (1 + 2 \cos \theta)^3 \sin \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{2}} (1 + 2 \cos \theta)^3 \sin \theta \frac{d\theta}{dt} dt \\
 &= \int_1^0 (1 + 2t)^3 (-1) dt \\
 &= \left[\frac{(1 + 2t)^4}{2 \times 4} \right]_1^0 \\
 &= \frac{81}{8} - \frac{1}{8} = 10.
 \end{aligned}$$

Exercises

Evaluate the following:

1. $\int_0^{\frac{\pi}{2}} \frac{\sin \theta}{1 + \cos^2 \theta} d\theta$ 2. $\int_0^1 \frac{(\tan^{-1} x)^2}{1 + x^2} dx$ 3. $\int_0^1 \frac{\sqrt{\tan^{-1} x}}{1 + x^2} dx$

Ans. 1. $\frac{\pi}{4}$ 2. $\frac{\pi^3}{192}$ 3. $\frac{\pi\sqrt{\pi}}{12}$

50 To show that $\int_0^a f(x) dx = \int_0^a f(a - x) dx$.

Put $x = a - t$, so that $\frac{dx}{dt} = -1$.

When $x = 0$, $t = a$ and when $x = a$, $t = 0$.

$$\begin{aligned}
 \therefore \int_0^a f(x) dx &= \int_a^0 f(x) \frac{dx}{dt} dt \\
 &= \int_a^0 f(a - t) (-1) dt \\
 &= \int_0^a f(a - t) dt \\
 &= \int_0^a f(a - x) dx, \text{ since a definite}
 \end{aligned}$$

integral is independent of the variable of integration.

Worked Example

Show that $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{4}$.

By Article 50, $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \sin^2 \left(\frac{\pi}{2} - x \right) dx$

$$= \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

Each of these equal integrals is therefore equal to half their sum.

Each $= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) dx$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= \frac{1}{2} (x)_0^{\frac{\pi}{2}} = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{4}.$$

Exercises

Show that the value of each of the following integrals

is $\frac{\pi}{4}$:

1. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$ 2. $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \tan x}$

$$3 \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$$

$$4 \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$5 \int_0^{\infty} \frac{x dx}{(1+x)(1+x^2)}$$

Evaluate the following:

$$6 \int_0^{\infty} \frac{dx}{1+x^2}$$

$$7 \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

$$8 \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$9 \int_0^4 \frac{dx}{x^2+16}$$

$$10 \int_0^{\frac{\pi}{2}} \sin^3 x \cos x dx$$

$$11 \int_0^{\pi} x \cos x dx$$

$$12 \int_0^{\frac{\pi}{2}} \sin^3 x dx$$

$$13 \int_0^{\frac{\pi}{2}} \cos^3 x dx$$

$$14 \int_0^{\frac{\pi}{2}} x^2 \sin x dx$$

Ans. 6 $\frac{\pi}{2}$ 7 π 8 $\frac{\pi}{2}$ 9 $\frac{\pi}{16}$ 10 $\frac{1}{4}$

11 -2 12 $\frac{2}{3}$ 13 $\frac{2}{3}$ 14 $\pi - 2$

APPENDIX

Second Year Pre-University Examination
Question Papers (Karnataka)

APRIL 1975

TIME: 3 HOURS]

[MAXIMUM MARKS—100

N. B.—(1) Answer *eight* full questions choosing at least one question from each section.

(2) All questions carry *equal* marks.

SECTION A

1 (a) Prove that the number of permutations of n things taken all at a time when p things are alike of one kind and q things alike of another kind and the rest are all different is $\frac{n!}{p! q!}$.

(b) Find the number of permutations of the letters of the word "MISSISSIPPI". How many of them begin with M?

(c) Resolve into partial fractions:—

$$\frac{2x+5}{(x+2)(x-1)^2}$$

2 (a) Prove by Mathematical Induction:—

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

(b) Find the middle terms in the expansion of

$$\left(\sqrt{x} - \frac{5}{x^2} \right)^{11}$$

(c) With the usual notation, prove that

$$c_0 + 2c_1 + 3c_2 + 4c_3 + \dots + (n+1)c_n = 2^{n-1}(n+2).$$

3 (a) $2A + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ and $3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$, find A and B .

(b) Prove that a determinant is unaltered in value if to the elements of a column same multiples of the corresponding elements of any other column are added.

(c) Show that
$$\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc$$

SECTION B

4 (a) In any triangle ABC , prove that $a^2 = b^2 + c^2 - 2bc \cos A$.

(b) In triangle ABC , prove that

$$\frac{a+b}{c} = \frac{1 + \tan \frac{A}{2} \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}}$$

(c) If in the triangle ABC , $a \cos A = b \cos B$, prove that either the triangle is isosceles or right angled.

5 (a) Solve the triangle ABC , given

$$a = \sqrt{6} - \sqrt{2}, \quad b = 2\sqrt{2} \quad \text{and} \quad C = 120^\circ.$$

(b) Find the general solution of:—

$$\tan \theta + \tan 2\theta = \tan 3\theta$$

(c) Prove that $\sin^{-1} \frac{3}{5} + \cot^{-1} 7 = \frac{\pi}{4}$

6 (a) Find the equation to the line passing through the intersection of $3x - 4y - 2 = 0$ and $5x + 4y - 14 = 0$ and perpendicular to $x + 2y + 10 = 0$.

(b) Determine 'a' so that (2, 3) and (-4, a) may be at equal distances from the line $3x + 4y = 8$.

(c) Find the coordinates of the orthocentre of the triangle ABC , given $A(2, 1)$, $B(5, 2)$, and $C(3, 4)$.

7 (a) Find the equation of a circle described on the line joining the points (x_1, y_1) and (x_2, y_2) as diameter.

(b) Find the length of the tangent from $(6, -1)$ to the circle $x^2 + y^2 + 2x - 4y - 11 = 0$.

(c) Find the radical centre of the circles:—

$$x^2 + y^2 + 4x + 2y + 1 = 0.$$

$$2x^2 + 2y^2 + 8x + 6y - 3 = 0.$$

$$x^2 + y^2 + 6x - 2y - 3 = 0.$$

SECTION C

8 (a) Prove that $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1}$ for all rational values of n .

(b) Find (1) $\lim_{x \rightarrow 0} \frac{x}{1 - \sqrt{1-x}}$.

(2) $\lim_{\theta \rightarrow 0} \frac{\sin m\theta}{\sin n\theta}$

(c) Differentiate from first principles $\sin 2x$.

9 (a) Prove that $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$ where u and v are functions of x .

(b) Find $\frac{dy}{dx}$ if $x = a(\cos \theta + \theta \sin \theta)$
 $y = a(\sin \theta - \theta \cos \theta)$

(c) Given $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$.

10 (a) Prove that the curves $y = x^3$ and $6y = 7 - x^2$ intersect orthogonally at $(1, 1)$.

(b) If the equation of the tangent to the curve $y = 3x^2 - 4x$ at $(1, -1)$ is in the form $ax - y + b = 0$, find 'a' and 'b'.

(c) The volume of a sphere is increasing at 4π c.c./sec. Find the rate of increase of the radius and of the surface area when the volume is 288π c.c.

11 Integrate w. r. t. x :—

(1) $\int \frac{\sqrt{\sin^{-1} x}}{1 - x^2} dx$

(3) $\int \frac{1}{x^2 - 5x + 4} dx$

(2) $\int \sin 5x \cos 2x dx$

(4) $\int x \sin x dx$

12 (a) Evaluate:—

(1) $\int_0^{\pi} \sin^2 x dx$

(2) $\int_0^1 \frac{(\tan^{-1} x)^2}{1 + x^2} dx$

(3) $\int_0^{\pi} \sin 2x \cos x dx$

(b) Find the area bounded by the curve $y = x^2 - 7x + 10$ and the x -axis.

All questions carry equal marks

SECTION A

1. (a) Resolve into partial fractions;—

$$\frac{9}{(x-1)(x+2)^2}$$

- (b) Using the formula for
- ${}^nP_r$
- ,

$$\text{prove that } {}^nP_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}.$$

- (c) How many words of two consonants and one vowel can be formed from six consonants and three vowels, the vowel occupying the middle position?

2. (a) Prove by Mathematical Induction:—

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

- (b) Obtain the coefficient of
- x^{11}
- in the expansion of

$$\left(x + \frac{2}{x^2}\right)^{17}$$

- (c) Sum up the series:—

$$c_0 + 6c_1 + 11c_2 + \dots \text{ to } (n+1) \text{ terms.}$$

3. (a) If
- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$
- and
- $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$
- ,

prove that $AB \neq BA$.

- (b) If in a determinant, any two rows (or columns) are interchanged, the sign of the determinant is altered.

$$(c) \text{ Show that } \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

SECTION B

4. (a) In any
- $\triangle ABC$
- ,
-
- prove that
- $b^2 = c^2 + a^2 - 2ca \cos B$

- (b) Show that in
- $\triangle ABC$
- ,

$$\tan \frac{A-C}{2} = \frac{a-c}{a+c} \cot \frac{B}{2}$$

- (c) With the usual notation for
- $\triangle ABC$
- , prove that

$$\frac{a}{bc} + \frac{\cos A}{a} = \frac{b}{ca} + \frac{\cos B}{b} = \frac{c}{ab} + \frac{\cos C}{c}$$

5. (a) Solve
- $\triangle ABC: a=2, b=\sqrt{6}, c=\sqrt{3}+1$
- .

- (b) Find general solution of
- $\sin \theta + \cos \theta = 1$
- .

- (c) Prove that
- $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

6. (a) Obtain the angle between the straight lines
-
- $ax + by + c = 0$
- and
- $a^1x + b^1y + c^1 = 0$
- .

- (b) Find the equation of the straight line passing through the intersection of
- $2x - 5y = 1$
- and
- $3x + 2y = 8$
- and parallel to
- $2x + y = 3$
- .

- (c) What are the coordinates of the orthocentre of the triangle whose vertices are
- $(13, 2)$
- ,
- $(-5, 6)$
- ,
- $(-5, -7)$
- ?

7. (a) Derive an expression for the length of the tangent from the point
- (x_1, y_1)
- to the circle
-
- $x^2 + y^2 + 2gx + 2fy + c = 0$
- .

- (b) Show that the circles
- $x^2 + y^2 + 2gx + 2fy + c = 0$
- and
- $x^2 + y^2 + 2g^1x + 2f^1y + c^1 = 0$
- cut orthogonally if
- $2gg^1 + 2ff^1 = c + c^1$
- .

- (c) Find the equation of the circle passing through the origin, having its centre on
- $x + y = 4$
- and cuts orthogonally
- $x^2 + y^2 - 4x + 2y + 4 = 0$
- .

SECTION C

8. (a) Prove that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ (x in radians)
- (b) Find (i) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$
- (ii) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$
- (c) Obtain the differential coefficient of $\sin x$ from first principles.
9. (a) If u and v are functions of x , prove that
- $$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$
- (b) If $x = a \cos^3 t$, $y = b \sin^3 t$, evaluate $\frac{dy}{dx}$ when $t = \frac{\pi}{4}$.
- (c) Differentiate y with respect to x , if $y = (x^2 + 2x + 1)^5$, $\cos^3 x$
- (d) Find $\frac{dy}{dx}$, if $2x^2 + 3xy + 4y^2 = 1$
10. (a) Find the angle of intersection of the curves $x^2 + 2y^2 = 14x - 16$ and $y^2 = 2x$ at $(2, 2)$.
- (b) In the curve $x^3 y^2 = a^4$, show that the sub-tangent at any point varies as the abscissa of the point.
- (c) The distance s travelled by a particle in time t is given by $s = t^3 - 6t^2 - 4t - 8$. Find when the acceleration is zero. Also find the velocity at that instant.

11. Integrate with respect to x :-

- (a) $(x^2 + 2x + 3)^{\frac{1}{2}} (x + 1)$
- (b) $\sec^3 x \tan x$
- (c) $\sin^{-1} x$
- (d) $\frac{1}{4x^2 + 9}$

12. (a) Evaluate:-

- (i) $\int_0^1 (2x + 3)^5 dx$
- (ii) $\int_0^{\frac{\pi}{2}} \cos^2 x dx$
- (iii) $\int_0^{\pi} \sin 3x \sin x dx$.

(b) Find the area of the circle $x^2 + y^2 = a^2$ by integration.

APRIL 1976

All questions carry equal marks

SECTION A

1. (a) Resolve into partial fractions:— $\frac{x^2+1}{x(x^2+2)}$
- (b) 5 men and 5 women are to be seated round a table. In how many ways can they all be arranged if
 - (i) all the women are to sit together
 - (ii) no two women sit together
- (c) With the usual notation, prove that

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

2. (a) Prove by mathematical induction:—
 $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- (b) In the expansion of $(1+x)^{43}$, the coefficient of the $(2m+1)^{th}$ and $(m+2)^{nd}$ terms are equal. find m .

- (c) With the usual notation, prove that
 $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

3. (a) If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$,
find AB' and $A'B$

- (b) Show that the value of a determinant is unaltered if rows and columns are interchanged.

- (c) Solve, using Cramer's rule: $7x+6y-5z=30$,
 $3x-4y+z=0$, $x+2y-3z=10$

SECTION B

4. (a) In any triangle ABC , prove that
 $a = b \cos C + c \cos B$

- (b) In a triangle ABC , given $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$,
show that the triangle ABC is equilateral.

- (c) In triangle ABC , prove that
 $\frac{\cos 2A}{a^2} + \frac{\cos 2B}{b^2} = \frac{1}{a^2} + \frac{1}{b^2}$

5. (a) Solve the triangle ABC :—
 $b = \sqrt{3}$, $c = 1$ and $A = 30^\circ$

- (b) Find the general solution of:—
 $\sin \theta + \sin 2\theta + \sin 3\theta = 0$

- (c) Prove that, $\tan^{-1} \frac{3}{5} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{4}{7}$

6. (a) Derive an expression for the length of the perpendicular from (x_1, y_1) to the straight line $ax + by + c = 0$

- (b) Find the angle between the lines
 $3x - 4y + 5 = 0$, $x + 2y - 8 = 0$

- (c) Find K so that the lines $x + y = 3$, $2x + y = 5$ and $Kx + 3y = 11$ may be concurrent.

7. (a) Find the equation of the circle described on the line joining the points (x_1, y_1) and (x_2, y_2) as diameter.

- (b) Find the value of K , if the straight line $x + Ky - 5 = 0$ touches the circle
 $x^2 + y^2 - 2x - 6y - 6 = 0$.

- (c) Find the radical centre of the Circles:
 $x^2 + y^2 + 6x + y - 9 = 0$, $x^2 + y^2 + 4x - 9 = 0$,
 $x^2 + y^2 + 4x + 2y = 5$

SECTION C

8. (a) Find (i) $\text{Lt}_{x \rightarrow -2} \frac{x^5 + 32}{x + 2}$ (ii) $\text{Lt}_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x)$

$$(iii) \text{Lt}_{n \rightarrow \infty} \frac{1^3 + 2^3 + 3^3 + \dots + n^3}{n^4}$$

- (b) Differentiate from first principles $\sec x$
9. (a) Differentiate the following with respect to x

(1) $\frac{(x+2) \cdot \sin x}{x^2 + 1}$ (2) $\tan^{-1}(\sin x)$

(b) Find $\frac{dy}{dx}$, if $x = \tan t$, $y = \sin t \cdot \cos t$

(c) Given $x^2 y^2 = x^2 + y^2$, find $\frac{dy}{dx}$.

10. (a) Find the equation of the tangent to the curve $y^2 = 3x + 2$ which is parallel to the line $3x - 4y + 1 = 0$.

- (b) Prove that in the curve $xy = c^2$, the Sub-normal varies as the cube of the ordinate.

- (c) Air is being blown into a foot-ball bladder at the rate of 72 cubic cms./Sec. Find how fast the radius is increasing when the radius is 8 cms.

11. Integrate with respect to x :-

(i) $\frac{1}{4x^2 + 4x + 5}$ (ii) $\frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}$

(iii) $\sin 5x \cdot \cos 2x$ (iv) $x \cdot \sin 3x$

12. (a) Evaluate:-

(i) $\int_0^{\frac{\pi}{2}} \sin^3 x \cdot dx$ (ii) $\int_0^{\frac{\pi}{8}} \sec 2x \cdot \tan 2x \cdot dx$

(iii) $\int_0^1 (x^2 + 1)^2 \cdot dx$

- (b) Find the area enclosed by the curve $y = x^2 + 2x + 3$, x -axis and ordinates at $x = 1$; $x = 5$.

OCTOBER 1976

All questions carry equal marks

SECTION A

1. (a) Resolve into partial fractions: $\frac{3x^2 + 2x - 2}{(x-1)^2(2x-1)}$
- (b) How many numbers of four different digits can be formed using the digits 0, 1, 2, 3, 4, 5? How many of these numbers are odd?
- (c) With the usual notation, prove that ${}^nC_r = \frac{{}^nP_r}{r!}$

2. (a) Prove by mathematical induction:

$$2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 2$$

- (b) Find the term independent of x in the expansion of $\left(2x^2 - \frac{1}{x^2}\right)^8$

- (c) Sum up the series:

$$C_1 - 2C_2 + 3C_3 - \dots + n(-1)^{n-1}C_n$$

3. (a) Find x and y from the equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

- (b) Prove that the value of a determinant is not altered by adding to the elements of any row, the same multiples of the corresponding elements of any other row.

- (c) Prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

SECTION B

4. (a) In any triangle ABC , prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circumradius of the triangle.
- (b) In the triangle ABC , if $2\sin^2 B + 2\sin^2 C + \cos 2A = 1$, show that $\angle A = 90^\circ$
- (c) Prove that, in any triangle ABC , $a(b^2 + c^2) \cos A + b(c^2 + a^2) \cos B + c(a^2 + b^2) \cos C = 3abc$.
5. (a) Solve the triangle ABC , $\angle A = 45^\circ$, $\angle B = 105^\circ$ and $c = \sqrt{2}$
- (b) Find the general solution of; $\tan \theta - 2 \cot \theta = 1$
- (c) Prove that, $2 \tan^{-1} \left(\frac{1}{2} \right) + \cos^{-1} \left(\frac{4}{5} \right) = \frac{\pi}{2}$.
6. (a) Find the distance between the parallel lines $3x + 4y - 7 = 0$ and $6x + 8y - 9 = 0$.
- (b) Find the equation of the line through the intersection of the lines $3x + 5y + 7 = 0$, $2x + 3y + 4 = 0$ and parallel to x -axis.
- (c) Find the condition that the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ may be concurrent.
7. (a) Find the equation of the circle two of whose diameters are $x + y = 6$ and $x + 2y = 4$ and whose radius is 10 units.
- (b) Find the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at a given point (x_1, y_1) on it.
- (c) Show that two circles $x^2 + y^2 - 2x - 2y = 7$ and $3x^2 + 3y^2 - 8x + 29y = 0$ cut orthogonally.

SECTION C

8. (a) Prove that $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$, for all rational values of n .
- (b) Find (i) $\lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^2 + 2x - 3}$ (ii) $\lim_{\theta \rightarrow 0} \frac{1 - \cos 2\theta}{\theta}$
- (c) Differentiate from first principles $\cos x$.
9. (a) Differentiate the following w.r.t. x :-
- (1) $\sqrt{\frac{1+x}{1-x}}$ (2) $(1+x^2) \tan^{-1} x$.
- (b) Find $\frac{dy}{dx}$, if $x = a(t - \sin t)$, $y = a(1 - \cos t)$.
- (c) If $a \cos mx + b \sin ny = 0$, find $\frac{dy}{dx}$.
10. (a) For the curve $x^2 + y^2 - 4x - 2y + 1 = 0$, find the equation of the tangent at $(2, -1)$ on it.
- (b) Derive expressions for the Sub-tangent and Sub-normal for the curve $y = f(x)$.
- (c) The height of a cone is 30 cms. The height of the cone is constant and the radius of the base is increasing at the rate of 0.25 cm/sec. Find the rate of increase of volume of the cone when the radius of the base is 10 cms.
11. (a) Integrate with respect to x ;
- (1) $\frac{x^3 + x^2 + 4}{\sqrt{x}}$
- (2) $\frac{1}{1 + \sin x}$ (3) $\sqrt{\tan x} \cdot \sec^2 x$. (4) $x \cdot \tan^{-1} x$.
12. (a) Evaluate;
- (1) $\int_0^1 \frac{dx}{x^2 + 2x + 5}$
- (2) $\int_0^{\frac{\pi}{2}} \sin 3x \cdot \cos x \cdot dx$ (3) $\int_0^{\pi} x \cdot \sin x \cdot dx$
- (b) Find the area enclosed between the Parabola $y^2 = 4x$ and the straight line $y = x$.

APRIL 1977

All questions carry equal marks

SECTION A

1. (a) Resolve into partial fractions:—

$$\frac{2x^2 - 7x + 1}{x^2 - 3x - 4}$$

- (b) There are 4 red balls and 6 white balls. In how many ways can 4 balls be chosen so that there shall be at least one red ball and one white ball in each selection?

- (c) With the usual notation prove that

$${}^n p_r = \frac{n!}{(n-r)!}$$

2. (a) Prove by mathematical induction:

$$1^2 + 3^2 + 5^2 + \dots \text{ to } n \text{ terms} = \frac{n(4n^2 - 1)}{3}$$

- (b) Using the binomial theorem simplify

$$(2 + \sqrt{3})^5 + (2 - \sqrt{3})^5.$$

- (c) With the usual notation, prove that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}.$$

3. (a) If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

Show that $(AB)' = B'A'$.

- (b) If in a determinant, any two rows (columns) are identical, prove that the determinant vanishes.

- (c) Evaluate

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

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11. Integrate w. r. t. x :

(i) $\frac{1}{\sqrt{5 + 4x - x^2}}$

(iii) $\frac{\cos x}{\sqrt{1 - 9 \sin x}}$

(ii) $\sec 4x \tan 4x$

(iv) $x \cos^2 x$

12. (a) Evaluate:

(i) $\int_0^a \sqrt{a^2 - x^2} dx$

(ii) $\int_0^\pi \sin \frac{x}{2} \cos \frac{x}{2} dx$

(iii) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

- (b) Find the area enclosed between the curve $y^2 = 4x$ and the straight line $x = 2y$.

OCTOBER 1977

Section A

1. (a) Resolve into partial fractions:

$$\frac{2x^2 - 1}{(x-2)(x^2 - 2x - 15)}$$

- (b) There are 6 bowlers and 9 batsmen in a cricket club. In how many ways can a team of 11 cricketers be selected so that the team contains

(i) exactly 4 bowlers; (ii) at least 4 bowlers.

- (c) Prove that ${}_nC_r + {}_nC_{r-1} = (n+1)C_r$.

2. (a) Prove by mathematical induction:

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

- (b) Find the coefficient of y^3 in the expansion of

$$\left(7y^2 - \frac{2}{y}\right)^{12}.$$

- (c) With the usual notation prove that

$$c_0 + \frac{c_2}{3} + \frac{c_4}{5} + \frac{c_6}{7} + \dots = \frac{2^n}{n+1}.$$

3. (a) If $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$, prove that

$$A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}.$$

Find also AA' .

- (b) Prove that a determinant can be expressed as a sum of two determinants if each element of a row of the determinant is expressed as the sum of two terms.

- (c) Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = (a-b)(b-c)(c-a).$$

Section B

4. (a) In any triangle ABC prove that
 $c^2 = a^2 + b^2 - 2ab \cos C$.

- (b) In $\triangle ABC$, prove that

$$\frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0.$$

- (c) In $\triangle ABC$, if $3a = b + c$ prove that

$$\sin \frac{A}{2} = \sin \frac{B}{2} \cdot \sin \frac{C}{2}.$$

5. (a) Solve the $\triangle ABC$ given
 $a = \sqrt{6}$, $b = 2$ and $A = 120^\circ$.

- (b) Prove that $\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{4}{7} + \tan^{-1} \frac{9}{7} = \frac{\pi}{2}$.

- (c) Find the general solution of
 $4 \sin^2 \theta - 8 \cos \theta + 1 = 0$.

6. (a) Find the length of the perpendicular from (x_1, y_1) to the straight line $ax + by + c = 0$.

- (b) Find the equations to the straight lines bisecting the angle between
 $3x + 4y = 5$ and $12x - 5y = 41$.

- (c) Find the equations of the straight lines passing through the point of intersection of $2x + y = 0$ and $x + y + 2 = 0$ and (i) parallel, (ii) perpendicular to $x - 6y + 11 = 0$.

7. (a) Derive the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at point $A(x_1, y_1)$ on it.

- (b) Show that the line $4x + 5y - 39 = 0$ is a tangent to the circle $x^2 + y^2 + 6x - 4y - 28 = 0$. Find also the coordinates of the point of contact.

- (c) Find the equation of the circle which cuts orthogonally the circles $x^2 + y^2 - 4x - 6y + 11 = 0$ and $x^2 + y^2 - 10x - 4y + 21 = 0$, and has its centre on $2x + 3y - 7 = 0$.

Section C

3. (a) Show that

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1} \text{ for positive rational values of } n.$$

- (b) Evaluate:

(i) $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta \cdot \sin \theta}$

(ii) $\lim_{n \rightarrow \infty} \frac{2n^2 - 7n + 11}{7n^2 - 4n + 19}$

- (c) Differentiate $\sec x$ from first principles.

9. (a) If u and v are functions of x , show that

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}.$$

- (b) Find $\frac{dy}{dx}$ for any two of the following:

(i) $y = \frac{x+1}{\sqrt{x^2+4}}.$

(ii) $y = \sin^{-1}(\tan x).$

(iii) $x = \sin^2 t; y = \sin t \cos t.$

- (c) If $x^3y^3 = x + y$, find $\frac{dy}{dx}.$

10. (a) Find the angle of intersection of the circles $x^2 + y^2 = 13$ and $x^2 + y^2 - 8x + 3 = 0$ at the point of intersection (2, 3).

- (b) Find the equations to the tangent and normal to the curve $2x^2 - xy + 3y^2 = 18$ at the point (3, 1) on it.

- (c) A spherical ball is being inflated at the rate of 35 c.c. per minute. How fast is the surface area increasing when the diameter is 14 cms?

11. Integrate with respect to x :

(i) $\sin 5x \sin 9x.$

(ii) $\frac{1}{x^2 - 4x + 13}.$

(iii) $\sin x \cos^3 x.$

(iv) $x \sin 4x.$

12. (a) Evaluate:

(i) $\int_1^2 \left(3 - \frac{2}{x^2}\right)^2 dx.$

(ii) $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x dx.$ (iii) $\int_0^{\frac{\pi}{2}} \cos^3 x dx.$

- (b) Find the area enclosed between the curve $y = 11x - 24 - x^2$ and the line $y = x.$

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Section A

1. (a) Resolve into partial fractions:

$$\frac{2x^2 - 7x + 1}{x^2 - 3x - 4}.$$

- (b) With the usual notation, prove that

$${}^n P_r = \frac{(n)!}{(n-r)!}.$$

- (c) From 7 gentlemen and 4 ladies a committee of 6 is to be formed. In how many ways can this be done when the committee contains (i) exactly 2 ladies; (ii) at least 2 ladies.

2. (a) State the principle of mathematical induction and use it to prove that

$$1 + 3 + 5 + \dots + (2n-1) = n^2.$$

- (b) Find the coefficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}.$

(c) With the usual notation show that

$$\frac{C_1}{C_0} + 2 \frac{C_2}{C_1} + \dots + n \frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$$

3. (a) If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$,

show that $(AB)' = B'A'$.

(b) Prove that the value of a determinant is not altered by adding to the elements of any row the same multiples of the corresponding elements of any other row.

(c) Solve by Cramer's rule:

$$x + y + z = 7$$

$$2x + 3y + 2z = 17$$

$$4x + 9y + z = 37$$

Section B

4. (a) Prove that in any triangle ABC
 $a^2 = b^2 + c^2 - 2bc \cos A$.

(b) If in $\triangle ABC$, $\sin^2 B + \sin^2 C = \sin^2 A$, then prove that $A = 90^\circ$.

(c) Show that in the $\triangle ABC$
 $a \sin(B-C) + b \sin(C-A) + c \sin(A-B) = 0$.

5. (a) Solve the $\triangle ABC$ given that
 $a=2$; $b=\sqrt{6}$ and $c=\sqrt{3}+1$.

(b) Show that $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$.

(c) Find the general solution of
 $\sin 5x + \sin 3x = 0$.

6. (a) Find the length of the perpendicular from $P(x_1, y_1)$ on to the line $ax + by + c = 0$.

(b) Find the equations of the bisectors of the

angles between the lines $12x - 5y + 7 = 0$ and $4x - 3y + 8 = 0$.

(c) Find the equation of the straight line passing through the point of intersection of $2x + 3y = 1$ and $3x - 4y = 6$ and parallel to $5x - y = 10$.

7. (a) Find the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) on it.

(b) Prove that $x - 2y + 1 = 0$ touches the circle $x^2 + y^2 + 3x - 2y + 2 = 0$.

(c) Find the value of c for which the circles $x^2 + y^2 - 2x + 22y + 3 = 0$ and $x^2 + y^2 + 14x + 6y + c = 0$ are orthogonal.

Section C

8. (a) If f and g are two functions of x given by $f(x) = x^2 + x + 1$ and $g(x) = 4x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

(b) Evaluate:

(i) $\lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1}{x}$

(ii) $\lim_{x \rightarrow \infty} \frac{3x^2 - 8x + 13}{10x^2 - 4x + 19}$

(c) Differentiate $\sin x$ with respect to x from first principles.

9. (a) If u and v are functions of x ,

prove that $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$.

(b) Differentiate the following with respect to x :

(i) $\sin 3x \cdot \cos^2 x$; (ii) $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$.

(c) If $x = a \cos^4 \theta$ and $y = b \sin^4 \theta$,

find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$.

10. (a) Find the angle of intersection of the curves $x^2 + 2y^2 = 14x - 16$ and $y^2 = 2x$ at $(2, 2)$.

(b) Show that at any point on the curve $ay^2 = (x+b)^3$ the subnormal varies as the square of the sub-tangent.

(c) If the distance s travelled in time t is given by $s^2 = at^2 + 2bt + c$, show that the acceleration varies inversely as s^3 .

11. Integrate with respect to x :

(i) $\frac{x^2}{x^2 + 9}$

(ii) $\tan^5 x \cdot \sec^2 x$

(iii) $x^2 \cos x$

(iv) $\frac{1}{\sqrt{5+4x-4x^2}}$

12. (a) Evaluate:

(i) $\int_1^2 \frac{x^2 - 3x + 2}{\sqrt{x}} dx.$

(ii) $\int_0^{\frac{\pi}{2}} \sin^3 x dx.$

(iii) $\int_0^1 \frac{\left(\tan^{-1} x\right)^3}{1+x^2} dx.$

(b) Find the area bounded by $y^2 = 4ax$, the X -axis and the ordinates $x=4$ and $x=8$.

attempt make question